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UNFOLDING THE DYNAMICS OF CREATIVITY AND INNOVATION

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A geometrical-boundary description of occurrence of novelty

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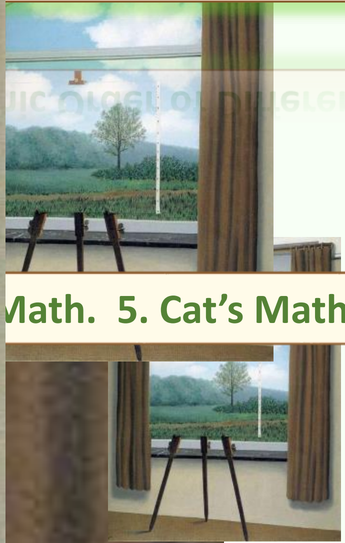


1. Introduction about … out novelty. Introducti

2. Geometry. Cohomology. … ohomology. Stokes

3. Dynamic Order of Differentiati … ic Order of Differentiation

4. Cat's Math. 4. Cat's M … Math. 5. Cat's Math. Cat's Math. 5. C

5. Applications. Heaps' and Zipf's Laws. Internet. 5. Applications. Heaps' and Zipf's Laws. Internet. 5.

- **Evolution, innovation, discovery and novelty are processes of high social, intellectual and practical importance.**

- **Novelty dynamics is enhanced by the explosion of publications, indexing, availability of documents, and one is feed-back for the other.**

- **This situation rapidly opened opportunities for developing the science of sciences, the quantitative understanding of the processes of building novelty, and of models for emergency of innovations, etc.**

During these developments some specific questions recurrently occurred:

1. **Is there a common scale invariant dynamics of discovery for various systems or sciences?**

2. **Can practice and success receipts be shared between different communities?**

Also, during these processes some ides and principles start to crystalize:

1. **For a system with boundary, each innovation "expands" or changes the boundary towards new types of future possibilities. The innovation can even change the very structure of the system's space. In that, the "surrounding" space of unexplored possibilities for a system (the adjacent possible) is changed continuously** [Kauffman, Tria, Loreto, Servedio, Strogatz, Buchanan, Castellano, Fortunato, *et al*]

2. **Topological transitions in systems' large scale structure of interaction (critical exponents, percolation). Self-similarity near critical point.** [Bettencourt, Kaiser, Castillo-Chavez, Vojick, Kaur, *et al*]

3. There are hierarchies of creativity: novelty, innovation, etc.

4. One cannot take the $\lim_{N\to\infty}$ limit, the systems are always finite, so statistical methods may be questionable.

5. Everything changes: size, number of constituents, boundary, space dimension, etc.

6. Nevertheless, systems that might have quite different local laws and dynamics behave similarly and self-similarly near a phase transition, via scaling.

7. There are triggering events, epidemics, long range interactions.

8. In spite of good dynamical similarity between a number of different systems, there are always other systems that have a different dynamics (publications=epidemics≠patents).

9. How constraints are built, and how new sources of free energy are detected?

10. How to select the useless accidents which will generate the future adjacent?

Geometric Boundary Approach

1. **Identify mechanisms to generate boundary** (interaction between constituents like in water surface, holographic principle, synchronization in molecular motors axonemal swimming)
2. **What types of boundary?**
3. **What types of change of boundary: intrinsic (shape), extrinsic (creation, change of dimension, change of topological properties).**
4. **The interplay between individual and surface.**
5. **Collective modes induced by surface. Can semantic be induced by boundary?**

Obvious examples of systems where boundary is very important

1. **Cell biology. Thermodynamic work to build the membrane, which in turn manipulates constraints on reactions, hence on constructing constraints that manipulate constraints**
2. **Heavy nuclei, neutron stars, supernovae core**
3. **Holographic principle in cosmology**
4. **Phase transition, criticality, self-similarity**

Less obvious examples of systems where boundary is very important

1. **The shape of internet (from the dynamics of interest and need of information)**
2. **Society**
3. **Cities**
4. **Invention patents**

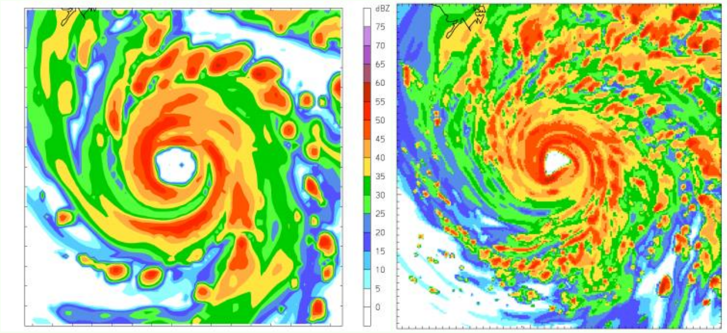
Engine of this study: The adjacent possible theory and the combinatorial (urn-like based) models for novelties:

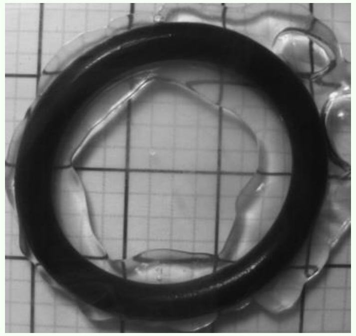
1. Geometry from graphs-networks: random walk on a growing graph whose structure is self-consistently shaped by the innovation process.

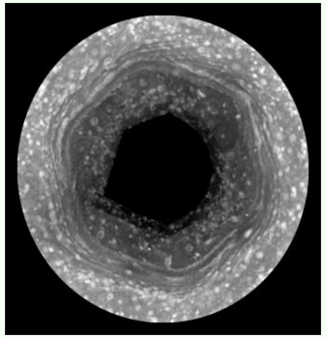
2. The notion of advance into the adjacent possible sets its own natural limits on innovations, since it implies that innovations too far ahead of their time, i.e., not adjacent to the current reality, cannot take hold. Careful where to set the boundary.

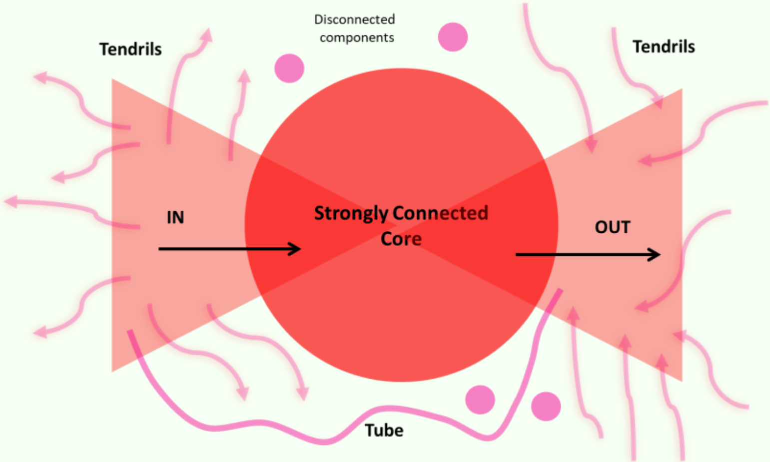
3. The fact that one observes the same statistical signatures for novelties and innovations strengthens the hypothesis that they could be manifestations of correlations generated by the expansion of the adjacent possible.

4. Showing an interesting dependence between the occurrence of novelty and the structure of boundaries of various spaces and adjacent possible structures.



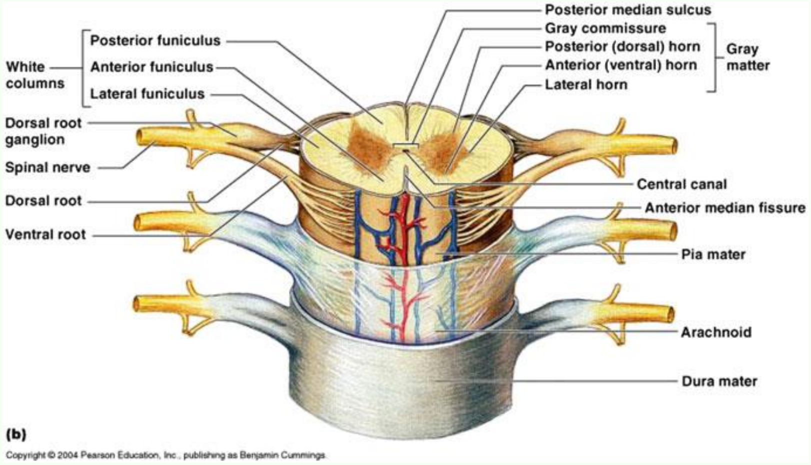






Internet model
2015 Broder, Kumar, Maghoul, Raghavan, Rajagopalaon, Stata, Tomkins, Wiener,

32×10^6



Spinal cord

Questions:

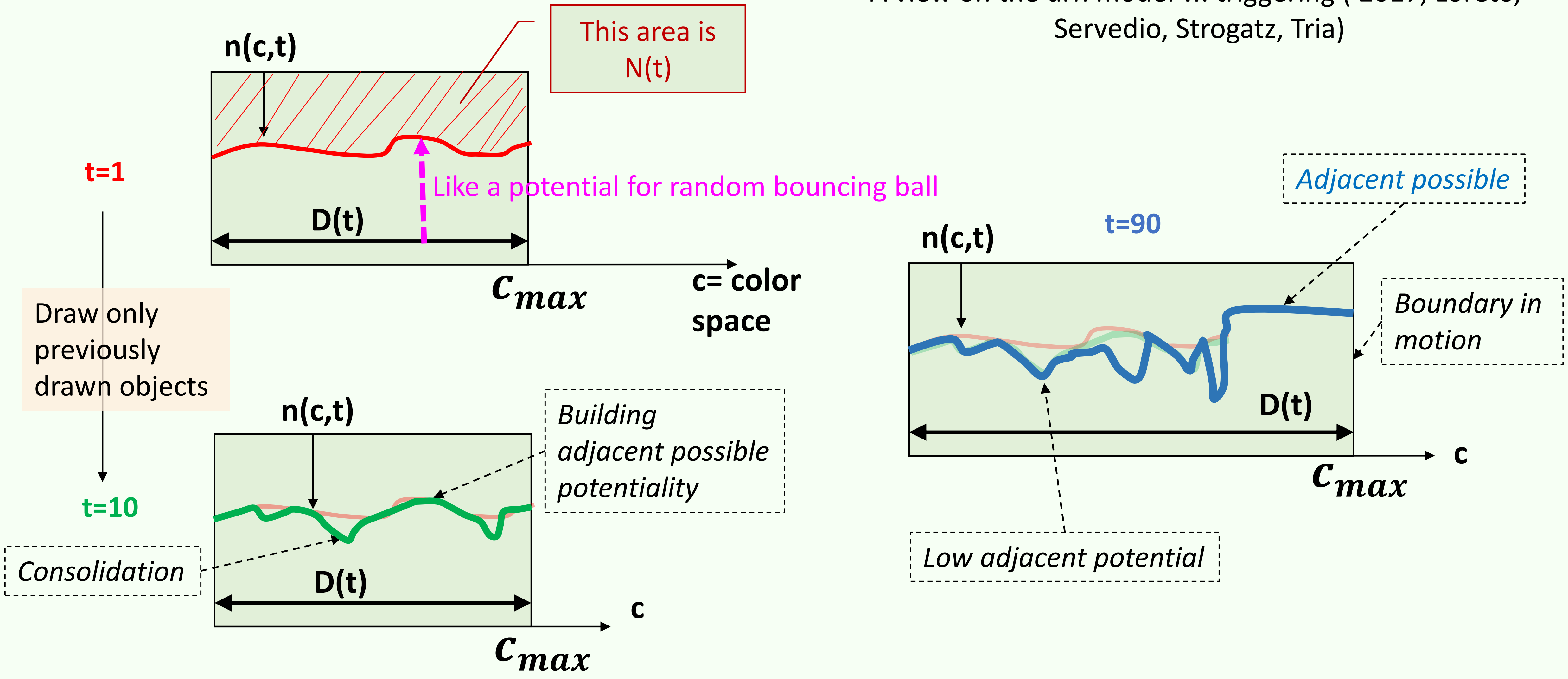
1. **How is the mathematics behind the dynamics of novelty? Can future of inventions and novelty, etc. can be really mathematically modeled?**

2. **If yes, do we use the present math tools, or may be a different approach would help even more?**

3. **Discovery is modeled: statistics, combinatorics, graphs/networks, criticality and percolation tools.**

4. **A smoother geometric approach may also help.**

A view on the urn model w. triggering (2017, Loreto, Servedio, Strogatz, Tria)



$$\left(\int_0^{c_{max}(t)} n(c,t)dc\right)^{\propto} = \|n^{\propto}\|_{fract} = \int_{\partial M} d\omega = |d\omega| \approx \omega(\partial M) \approx \|\partial M\|_{fract} \approx dia^{\frac{1}{\propto}} \approx$$

sub-linear power law in “radius”

Manifolds, simplexes and homology

M is n-dimensional differential manifold

K is its n-dimensional simplicial complex containing p-simplexes σ^p of dimension 0…n (triangulation of M)

$C_p(K)$ is the free abelian group generated by the oriented p-simplexes of K

$$c_p = \sum_{l=1}^{l_p} f_l \, \sigma_l{}^p, \qquad f_l \in Z$$

Boundary operator maps chains into smaller dimension chains

$\partial_p \colon C_p(K) \longrightarrow C_{p-1}(K)$, **by** $\partial_p[v_0, \dots, v_p] = \sum_{j=0}^{p} (-1)^j [v_0, \dots, \hat{v}_j \dots, v_p]$

Major theorem: $\partial_{p-1} \circ \partial_p = 0$ Boundary of a boundary is null.

In the following, by abuse of language, we write: K $\longrightarrow$ M

$Z_p(M)$ **is the group of p-dimensional cycles z_p in** $C_p(M)$ **such that $\partial z_p = 0$, that is the kernel of ∂.**

$B_p(M)$ **is the group of p-boundaries, b_p in** $C_p(M)$ **such that $b_p = \partial C_{p+1}$**

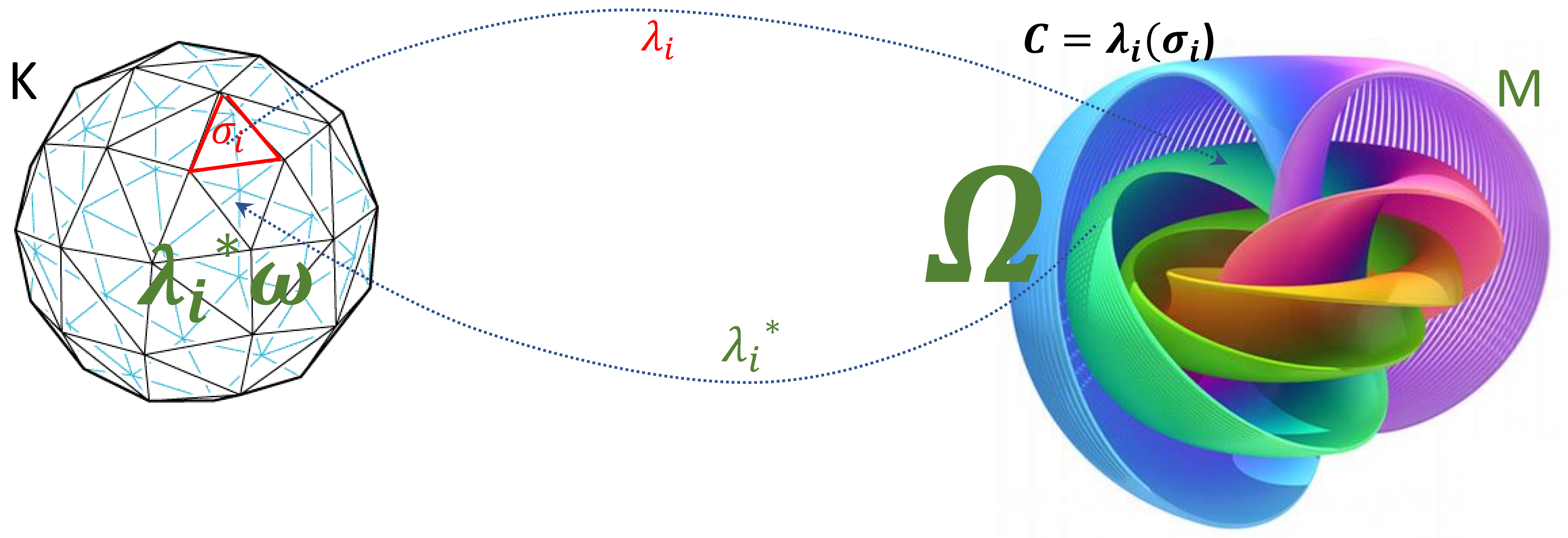
$H_p(M; \boldsymbol{R}) = Z_p(M)/B_p(M)$ **is the p-dimensional homology (quotient) group of M.**

Homology is independent of the triangulation K.

Examples: **The homology measures "how many holes and of what types has M"**

$\boldsymbol{H_p(S^n; R) = Z}$ **if $p = 0, n$ and $= \{0\}$ otherwise.**

$\boldsymbol{H_p(T^2; R) = Z}$ **if $p = 0, 2$ it is $= Z^2$ if p $= 1$ and $= \{0\}$ if p>3.**



$$\int_C \omega = \sum_{i=1}^{k} a_i \int_{\Delta_p} \lambda_i{}^* \omega \qquad \text{with} \qquad C = \sum_{i=1}^{k} a_i \lambda_i$$

de Rham Theorem in H. Whitney, *Geometric Integration Theory* (Princeton Univ. Press 1957)

Fundamental relation between space dimension and order of differentiation

Homology $\cdots \xleftarrow{\partial_{p-1}} C_{p-1} \xleftarrow{\partial_p} C_p \xleftarrow{\partial_{p+1}} C_{p+1} \xleftarrow{\partial_{p+2}} \cdots$

$$H_p(M;\boldsymbol{R}) = Z_p(M)/B_p(M) = \ker \partial_p \,/\, \operatorname{im} \partial_{p+1}$$

dual

Cohomology $\cdots \xrightarrow{d_{p-1}} \Omega^{p-1} \xrightarrow{d_p} \Omega^p \xrightarrow{d_{p+1}} \Omega^{p+1} \xleftrightarrow{d_{p+2}} \cdots$

$$H^p(M;\boldsymbol{R}) = Z^p(M)/B^p(M) = \ker d_{p+1} \,/\, \operatorname{im} d_p$$

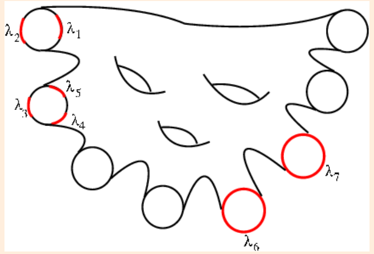
Cohomology involves Stokes formula:

$$\int_{M} d\omega = \int_{\partial M} \omega$$

For example: $$\iiint_{M} div\,\vec{V}\,dv = \oiint_{\partial M} \vec{V} \cdot d\vec{S}$$

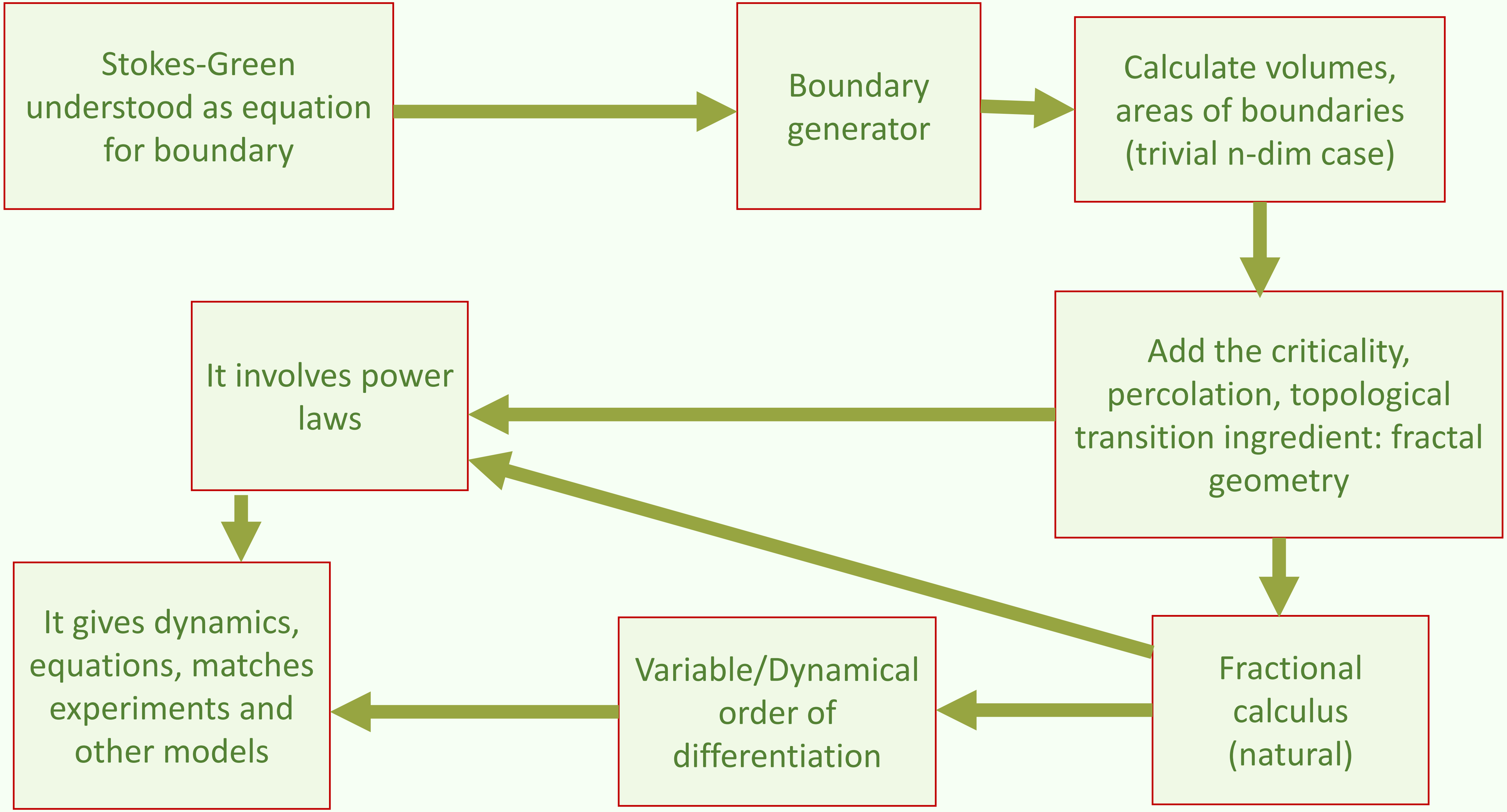
Present Stokes formula as an equation:

Given an m-dimensional manifold M, find a sub-manifold B of M of dimension m-1 such that for all m-1 forms ω on M we have:

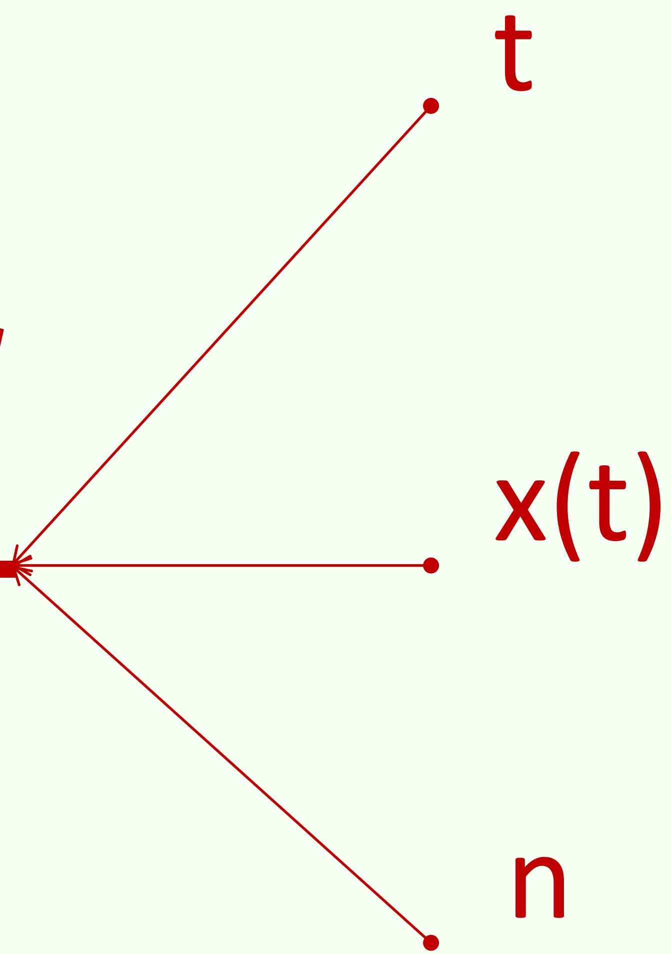


$$\int_{M} d\omega = \int_{B=?} \omega$$

If a solution exists and it's unique, B is the boundary of M, if it is not unique, and there exist q solutions, B is an elementary a cobordism of index q.


Stokes-Green understood as equation for boundary
Boundary generator
Calculate volumes, areas of boundaries (trivial n-dim case)
Add the criticality, percolation, topological transition ingredient: fractal geometry
It involves power laws
Fractional calculus (natural)
Variable/Dynamical order of differentiation
It gives dynamics, equations, matches experiments and other models

$$\frac{\partial^n x}{\partial t^n}$$



A model for occurrence of novelty in spaces of variable dimension by using the theory of time variable order of differentiation.

Equally, a natural frame for memory effects

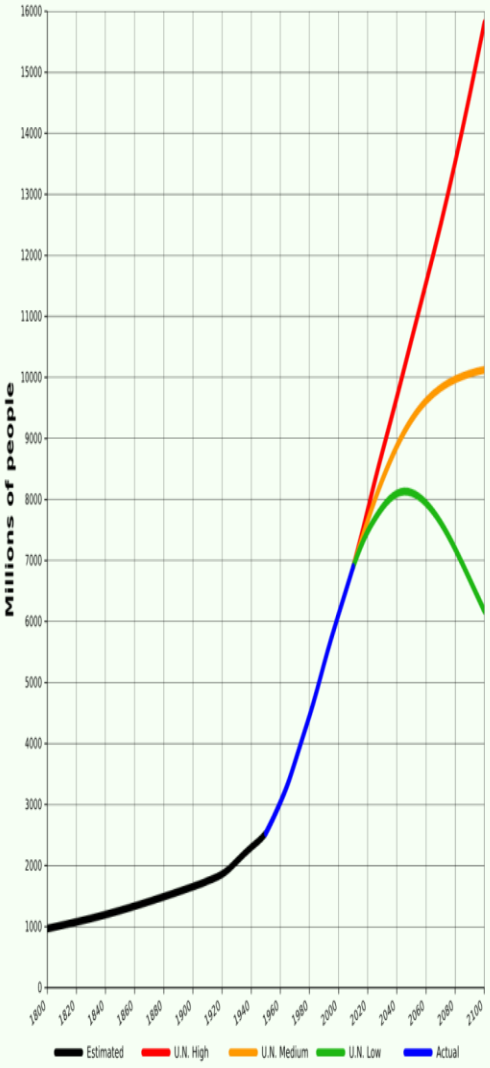
Beyond fractional

$\boldsymbol{Z}$ $\qquad$ $\boldsymbol{R}$ $\qquad$ $\boldsymbol{C}^{\infty}$ $\qquad$ $\mathcal{H}(q_1, q_2, p_1, p_2)$

$$\frac{\partial^n x}{\partial t^n} \rightarrow \frac{\partial^{\alpha} x}{\partial t^{\alpha}} \rightarrow \frac{\partial^{\alpha(t)} x}{\partial t^{\alpha(t)}} \rightarrow \frac{\partial^{q_1} q_2}{\partial t^{q_1}}$$

So how about this challenge:

$$\frac{\partial^t x}{\partial t^t}$$

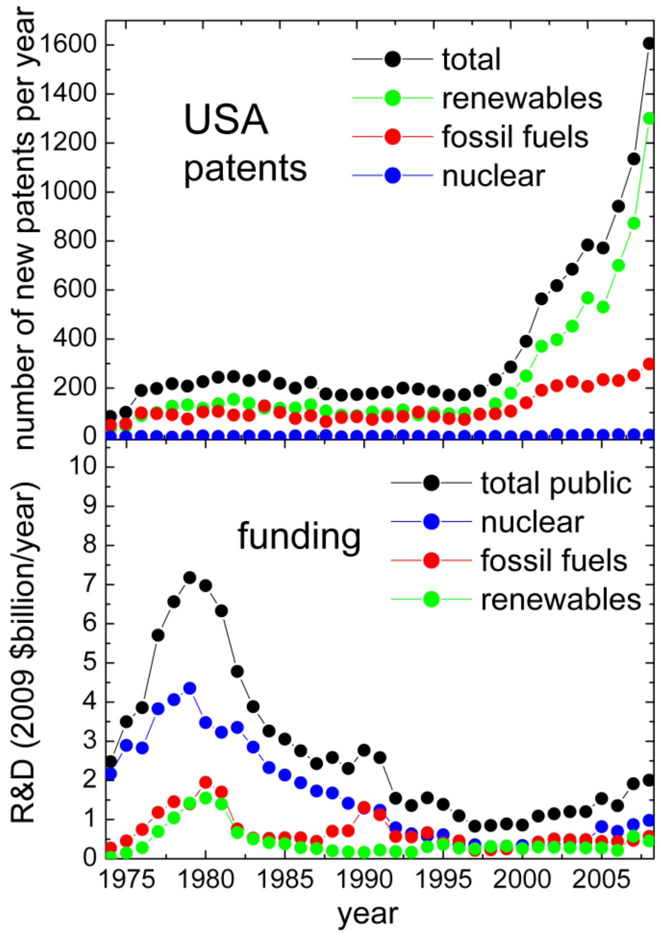


Examples of complex systems with time evolution from life and social systems to invention patents:

- Von Foerster (1960,) the Doomsday equation.
- Sergey Kapitsa (1965): $N = C \cot^{-1}\left(\frac{t_0 - t}{T}\right)$
- Hyperbolic growth models for population

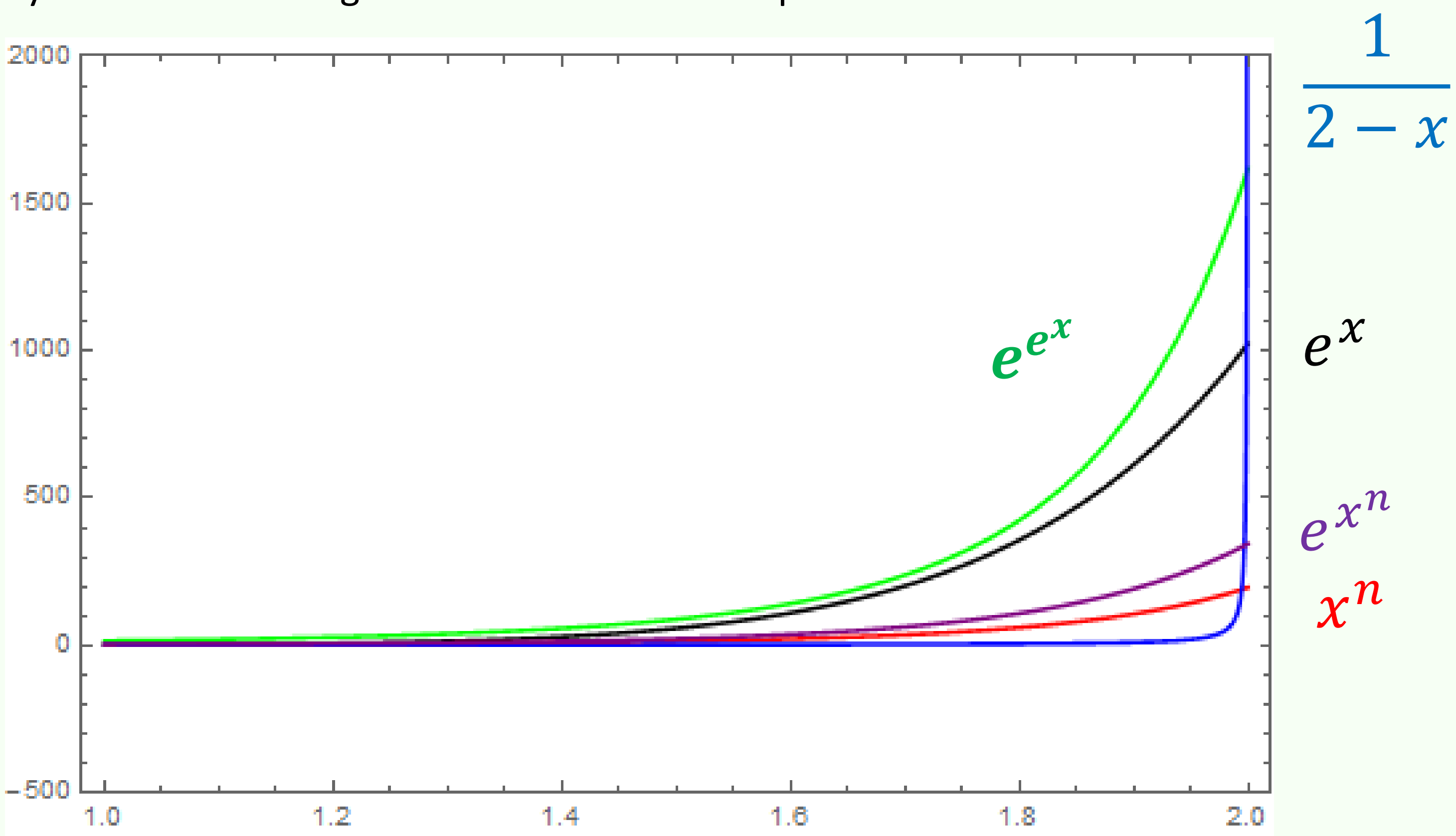
- Life is the most complex diversity spanning 21 orders of magnitude in size, obeying some empirical power laws (mass vs. metabolic rate, lifespan vs. heart rate, ¾ law, allometry, etc.)

- Gurevich and Varfolomeyev (2001): double exponential law for population growth:
  $N(t) = 375.6 \cdot 1.00185^{1.0073^{t-1000}}$
  with t in million years



<u>It appears that in real world the rapid growth is a transition from exponential and criticality</u>

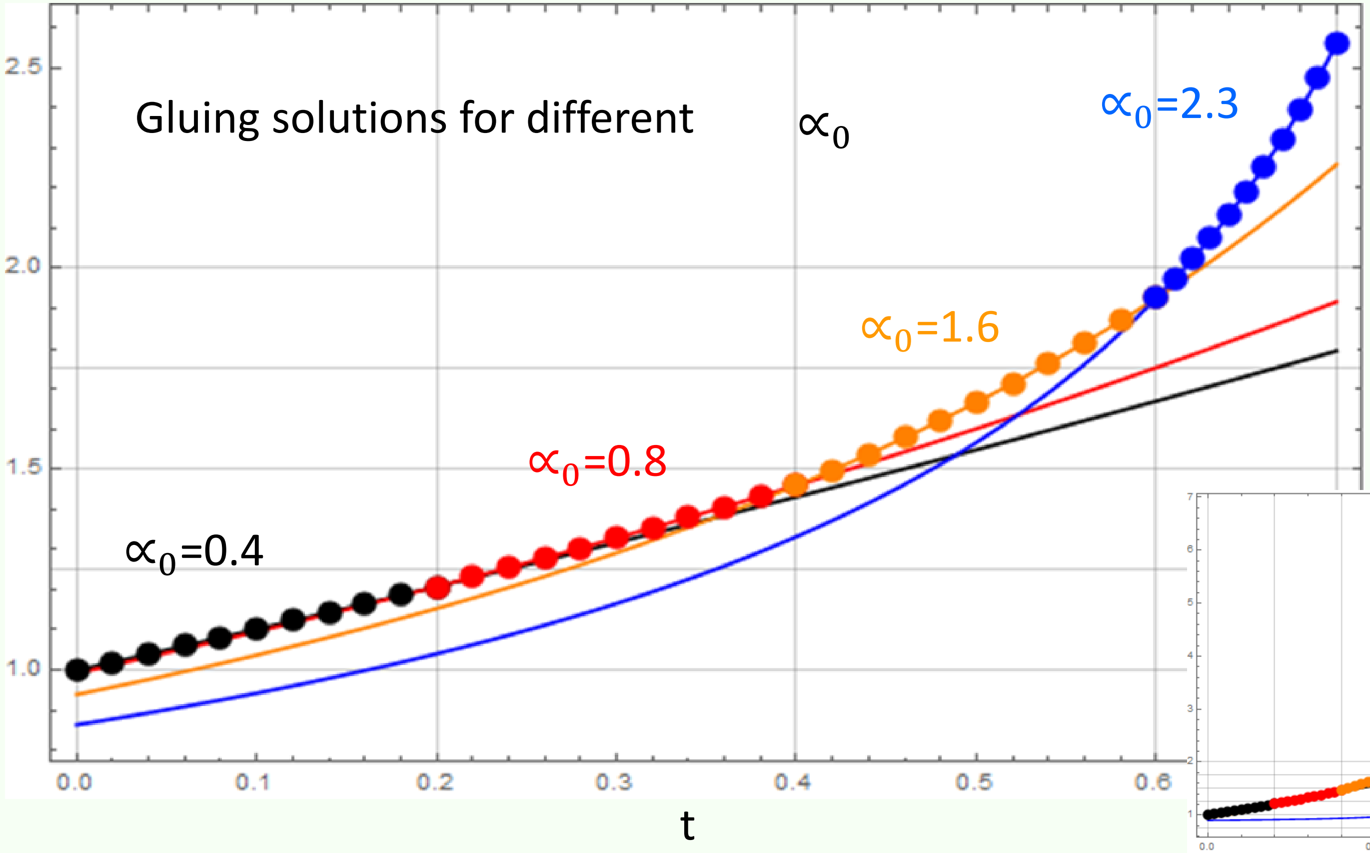
Many dynamical laws have exponential behavior over the long range, but critical phenomena are always faster in the neighborhood of the critical point (hyperbolic laws):



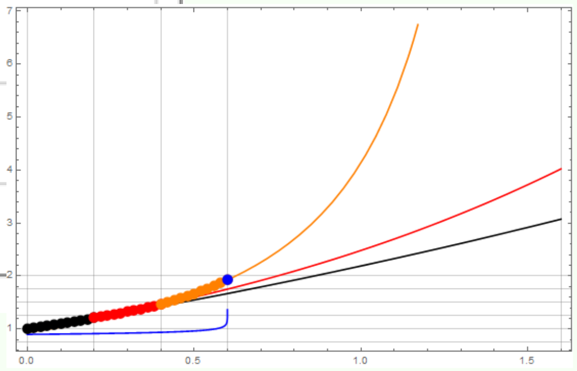
Constant exponent approximation

$$y' = y^{\propto_0},\ \text{y(0)} = y_0,\ \exists M > 0,\ \forall t \in \mathbf{I} \subset \mathbf{R},\ |\text{y(t)}|<\text{M}$$

$$y(t,\propto_0) = (\propto_0 - 1)\left(\frac{{y_0}^{1-\propto_0}}{\propto_0 - 1} - t\right)^{\frac{1}{1-\propto_0}}$$ It always has a singularity if $\propto_0 > 1$ at $t = \frac{\propto_0 - 1}{{y_0}^{1-\propto_0}}$



$$y(t,1) = y_0 e^t$$



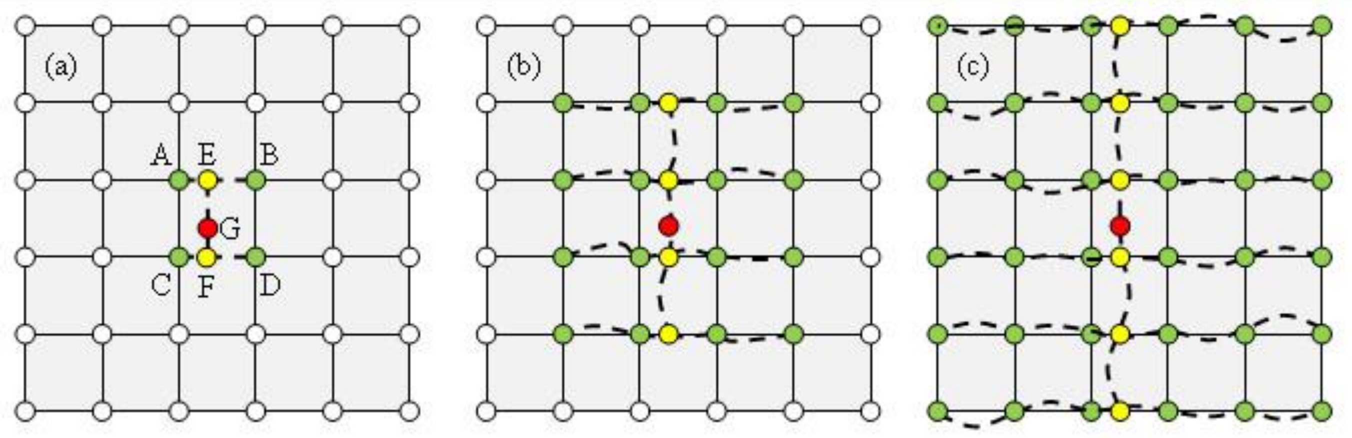
A way to change the dynamics:

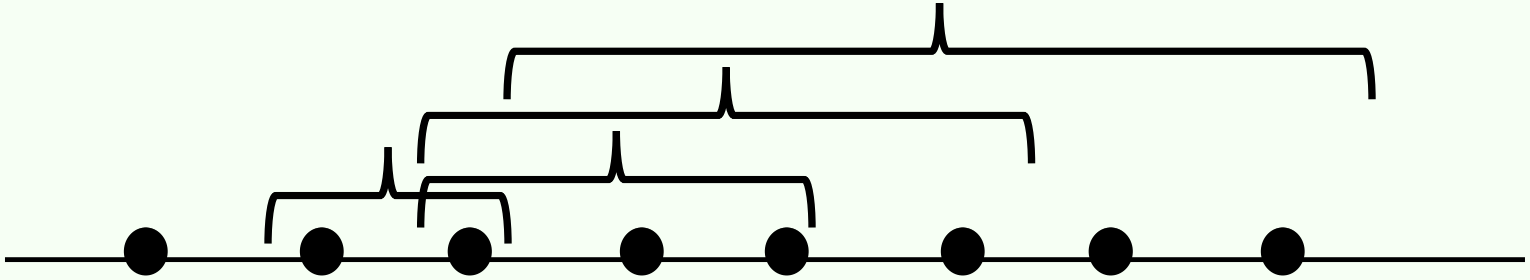
- **Dynamic order of differentiation (DODE)**
- **Advantages:**
  - **It keeps linearity (stability, linear spectral methods).**
  - **Some physical systems appear to follow such behavior (hydrodynamic drag, exploding wire, chaos, fractals, Levy motion, memory-dependent, porous flow, anomalous diffusion, Brownian systems, causal telegraph equation, battery recharging cycles, etc.)**
  - **Topological changes can occur when changing # of dimensions of space.**
  - **Contains memory, nonlocal and nonlinear effects.**

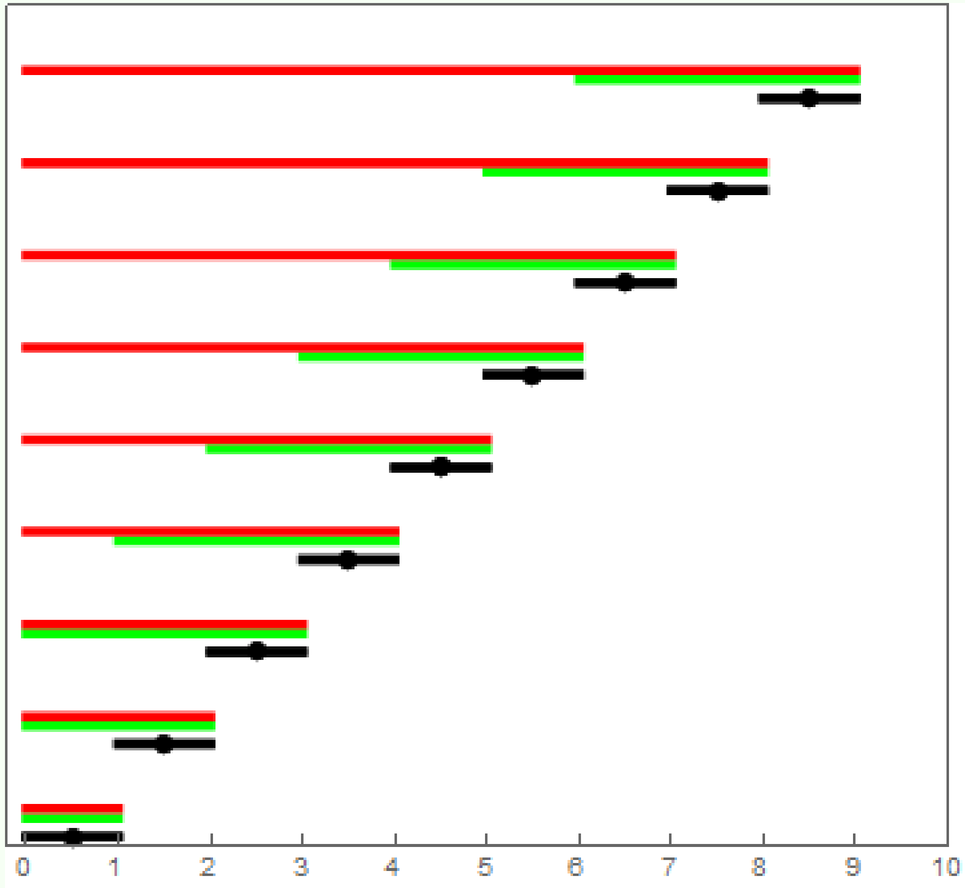
Higher order derivatives, finite difference and history dependence

$$y^{(n)} = \lim_{h \to 0} \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} y(x - kh)}{h^n}$$

Backward nth order finite difference







Finite difference equations of different orders:

- 1st order
- 4th order

Constant order finite differences are linearly history dependent

- Variable (increasing) order: the dynamics is also history dependent, but the amount of time taken into account from the past increases.

Example: human civilization has three types of memory genetic, neuronal, and external, involving different history ranges dependence and different scales in time.

A possibility to implement such equations would be fractional derivative ODE

$$De^{ax} = \frac{d}{dx}e^{ax} = ae^{ax} \qquad D^n e^{ax} = a^n e^{ax}$$

so

$$D^{1/2}e^{ax} = a^{1/2}e^{ax} \ ? \qquad D^{\propto}e^{ax} = a^{\propto}e^{ax} \ ? \qquad \text{for} \qquad \propto \in \boldsymbol{Q}, \boldsymbol{R}, \ldots \boldsymbol{C}$$

The meaning of integer order of differentiation:

$$\left.\begin{array}{l} e^{ax} = DD^{-1}e^{ax} \\ \\ e^{ax} = D\frac{1}{a}e^{ax} \end{array}\right\} \rightarrow D^{-1}e^{ax} = \int e^{ax}\,dx$$

in a formal way.

Generalizing:

$$D^{-2}e^{ax} = \iint e^{ax}\,dx\,, \qquad \dots \qquad D^{-n}e^{ax} = \underbrace{\iint \dots \int e^{ax}\,dx}_{\text{n-th iterated integral}}$$

If we generalize to rational number, we have technical questions:

1. Is this operator linear?
2. Does it obey composition law (closed)?
3. Is it correct to use antiderivatives?
4. What classes of functions can be fractionally differentiated like that?

More actions:

$$D^n x^p = \frac{p!}{(p-n)!} x^{p-n}$$

↓

$$D^{\propto} x^p = \frac{\Gamma(p+1)}{\Gamma(p-\propto+1)} x^{p-\propto}$$

There are problems:

$$D^{\propto} e^x \rightarrow e^x$$

$$D^{\propto} e^x \rightarrow D^{\propto} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{\Gamma(n+1) x^{n-\propto}}{\Gamma(n-\propto+1) n!}$$

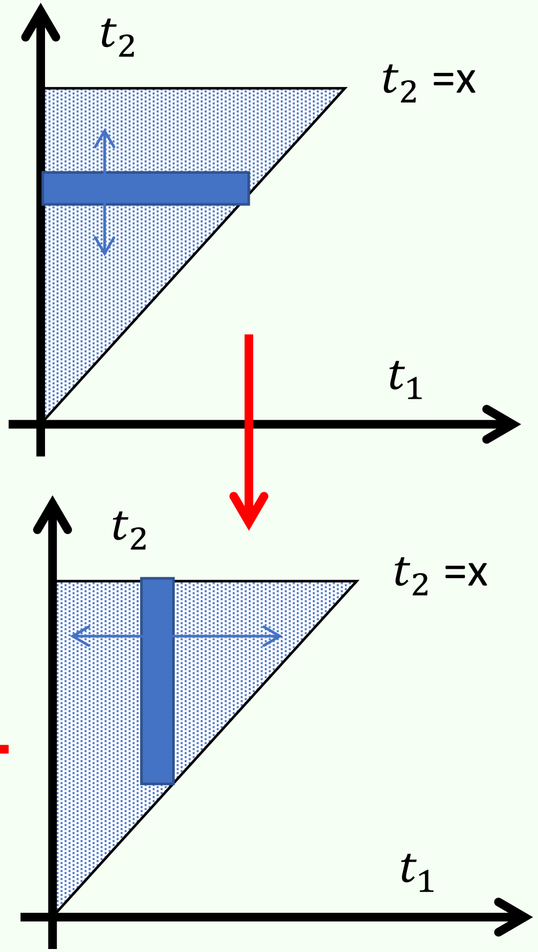
$\neq$

(unless $\propto \in \boldsymbol{Z}$)

This defect can be repaired by using geometrical insight for fractional derivative:

$$D^{-1}f(x) = \int f(x)\,dx \rightarrow \int_0^x f(t)dt$$

$$D^{-2}f(x) = \int_0^x\int_0^{t_2} f(t_1)\,dt_1\,dt_2 \rightarrow$$



$$D^{-2}f(x) = \int_0^x\int_{t_1}^x f(t_1)\,dt_1\,dt_2$$

$$= \int_0^x\int_{t_1}^x f(t_1)\,dt_2\,dt_1 = \int_0^x (x - t_1)f(t_1)\,dt_1$$

Now we can define: $$D^{-2}f(x) = \int_0^x (x-t)f(t)dt$$

and in general:

$$D^{-n}f(x) = \frac{1}{(n-1)!}\int_0^x f(t)\,(x-t)^{n-1}dt$$

and even more general: the fractional integral

$$D^{\propto}f(x) = \frac{1}{\Gamma(-\propto)}\int_0^x \frac{f(t)dt}{(x-t)^{\propto+1}}, \qquad \propto < -1$$

The fractional derivative is introduced by $\propto \in (0,1)$:

$$D^{\propto}f(x) = D^{\propto-m}D^m f(x) = D^{\propto-m}\frac{d^m}{dx^m}f(x), \quad m \geq 1,\ \propto - m < -1$$

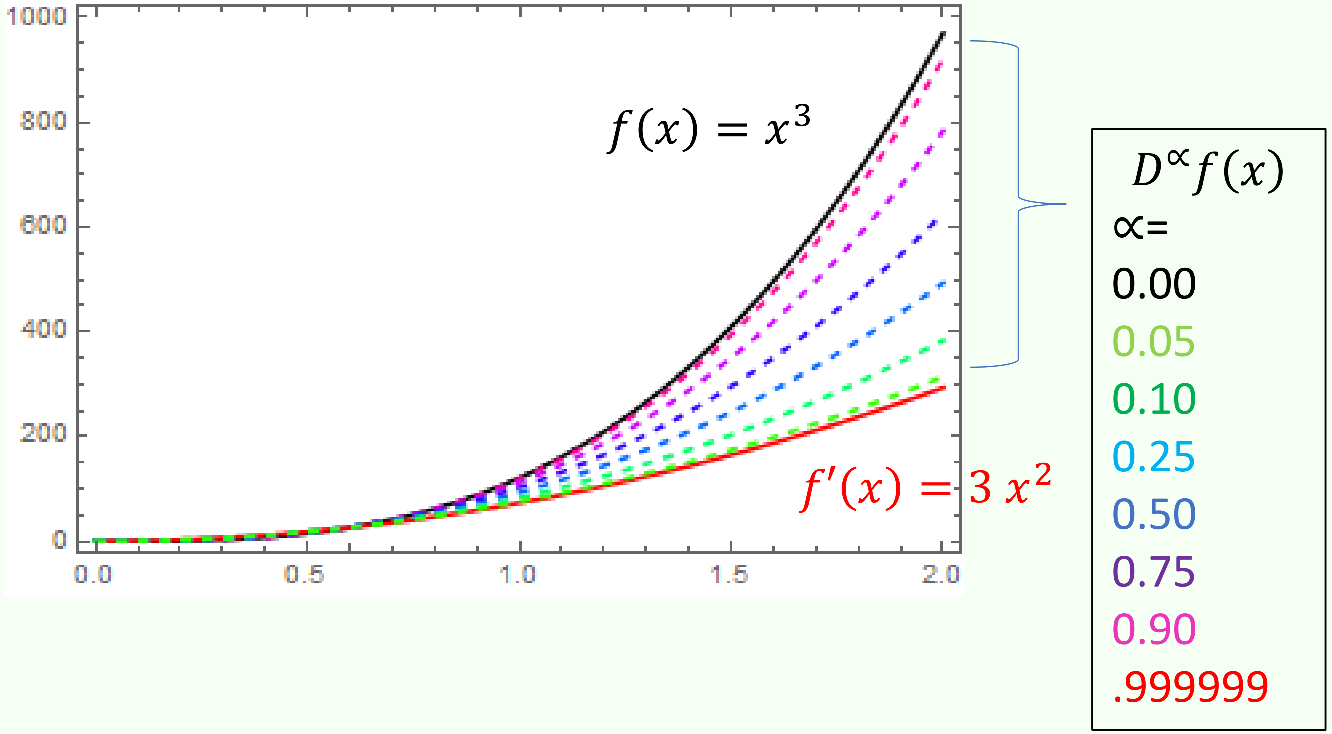
One possible definition is the Liouville-Riemann fractional derivative:

$${}_bD^{\propto}{}_x f(x) = \frac{d}{dx}\frac{1}{\Gamma(1-\propto)}\int_b^x \frac{f(t)dt}{(x-t)^{\propto}} \qquad \propto \epsilon(0,1)$$

We also have the Caputo fractional derivative:

$${}_bD^{\propto}{}_x f(x) = \frac{1}{\Gamma(1-\propto)}\int_b^x \frac{f'(t)dt}{(x-t)^{\propto}} \qquad \propto \epsilon(0,1)$$

How it works:


1000
800
600
400
200
0
0.0
0.5
1.0
1.5
2.0
$f(x) = x^3$
$f'(x) = 3\,x^2$
$D^{\propto} f(x)$
∝=
0.00
0.05
0.10
0.25
0.50
0.75
0.90
.999999

Let us start with the following example:

$$\frac{d^{\propto(t)}}{dt^{\propto(t)}} f + \varepsilon \frac{d}{dt} f = af$$

with $t \in I = [0, T], \quad a \in R,$
$0 < \varepsilon \ll 1, \quad \alpha(t) \in L_1(I) \to (0,1)$
$f(0) \neq 0$

We use for the variable order of differentiation a Riemann-Caputo fractional derivative of order $\propto(t)$, that is the ${}_0D^{\propto(t)}{}_t$

$$\frac{d}{dt} \frac{1}{\Gamma(1 - \propto(t))} \int_0^t \frac{f(s) - f(0)}{(t-s)^{\propto(t)}} dt + \varepsilon \frac{d}{dt} f = af$$

We integrate this equation once with respect to t:

$$\frac{1}{\Gamma(1-\propto(t))}\int_0^t \frac{f(s)-f(0)}{(t-s)^{\propto(t)}}dt + \varepsilon f = a\int_0^t f(s)ds + C$$

$$\int_0^t \left[\frac{1}{(t-s)^{\propto(t)}} - a\Gamma(1-\propto(t))\right] f(s)ds = f(0)\int_0^t \frac{ds}{(t-s)^{\propto(t)}} + C\Gamma(1-\propto(t))$$

$$-\varepsilon\,\Gamma(1-\propto(t))\ \mathrm{f(t)}$$

That is:

$$\boxed{\int_0^t K(t,s)f(s)ds + g(t) = f(t)}$$

So, we have a linear Volterra integral equation of 2nd kind

$$\int_0^t K(t,s)f(s)ds + g(t) = f(t)$$

with

$$g(t) = \frac{f(0)t^{1-\propto(t)}}{\varepsilon(1-\alpha(t))}$$

Continuous if $\alpha(t) \in (0,1)$

$$K(t,s) = (t-s)^{-\propto(t)} \underbrace{\left[-\frac{1}{\varepsilon\,\Gamma\big(1-\propto(t)\big)} + \frac{a}{\varepsilon}(t-s)^{\propto(t)}\right]}_{K_0(t,s)}$$

$K_0(t,s)$ continuous on $0 \le s \le t \le T$ and K(t,t)$\neq 0$

K(t,s) is a weakly singular kernel. This kernel is unbounded when s-t but Its integral over [0,T] is finite (integrable kernel).

This equation is also called Abel integral equation.

Theorem:

The Abel IE $\int_0^t K(t,s)f(s)ds + g(t) = f(t)$ with $g \in C^0(I), K_0 \in C^0(I)$

and weakly singular integrable kernel $K(t,s) = (t-s)^{-\propto(t)}\, K_0(t,s)$

with $\propto \in C^0(I)$ and $\exists \min_{t \in I} \propto(t) \neq 0$,

possesses a unique solution $f \in C^0(I)$.

This solution has the representation $f(t) = g(t) + \int_0^t R(t,s)g(s)ds$

where the resolvent kernel $R(t,s)$ of the kernel $K(t,s)$ has the form

$$R(t,s) = (t-s)^{-\propto(t)}\, Q_0(t,s)$$

with Q_0 continuous on D.

Theorem

If $\propto(t): R_+ \to (0,1)$ is continuous, and fulfils the condition that for all

$$p \in \left(1, \min_{t \geq 0}\left\{\frac{1}{\propto(t)}, \frac{1}{1-\propto(t)}\right\}\right) \text{ we have: } \sup_{t\geq 0} \frac{\Gamma(1+p(\propto(t)-1)}{\Gamma(\propto(t))} \leq +\infty$$

and $f(t,x): R_+ \times R \to R$ is continuous and fulfils:

$$|f(t,x) - f(t,y)| \leq F(t)|x-y| \quad \text{for all} \quad t \geq 0, \quad x, y \in R$$

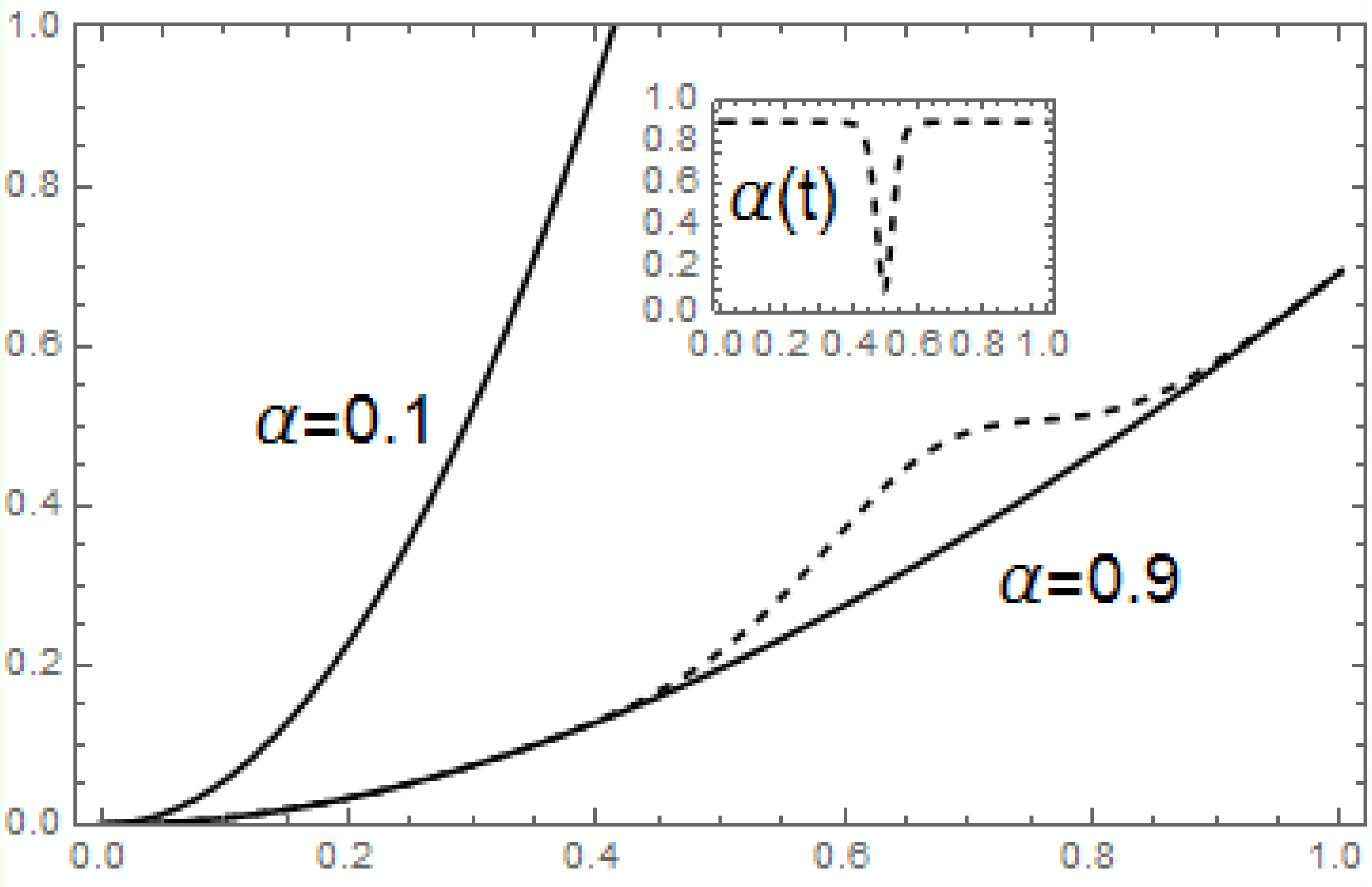
then the VODE initial condition problem:

$$D^{\propto(t)}(x - x_0) = f\big(t, x(t)\big), \qquad x(0) = x_0$$

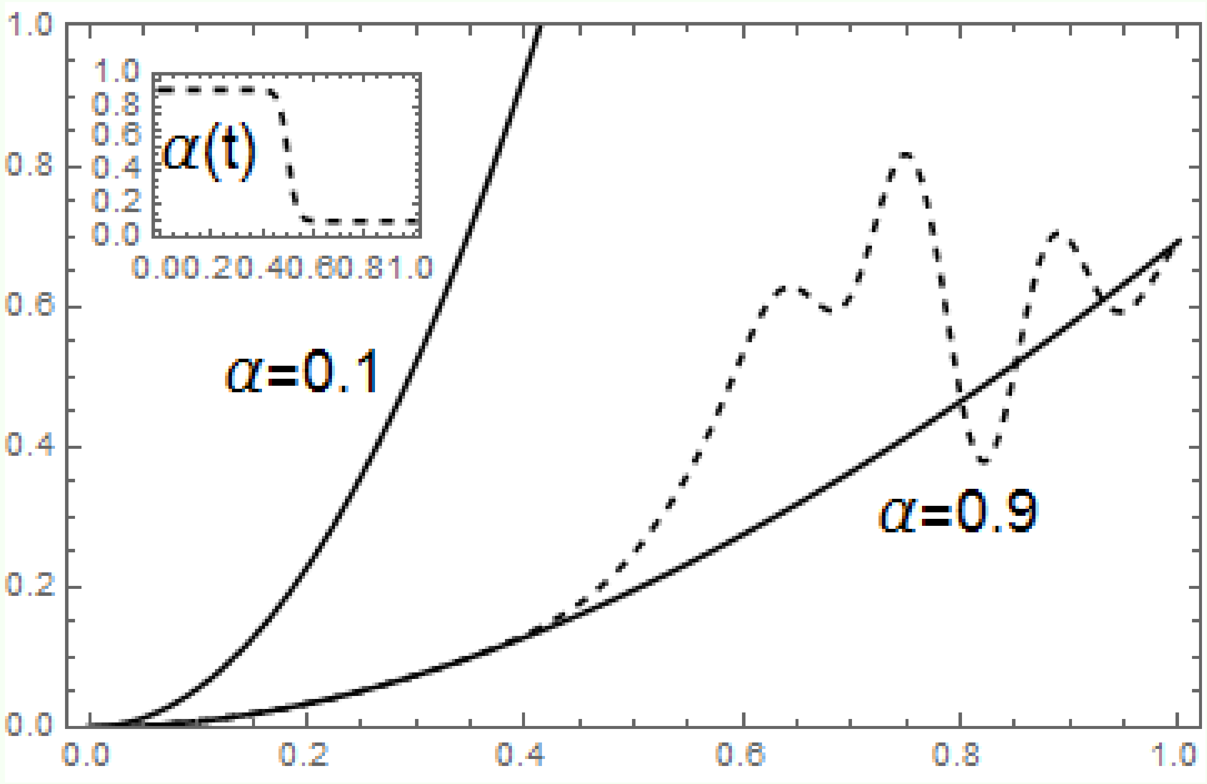
Has a unique solution on $t \geq 0$.

AL, Proc. AIP 9th Conf. Appl. Math. Techn. Natl. Sci., Albena, Bulgaria, vol. **1684** (2016) 323
AL, Proc. 11st AIMS Conf. on Part. Diff. Eqs., July 2016, Orlando, FL.

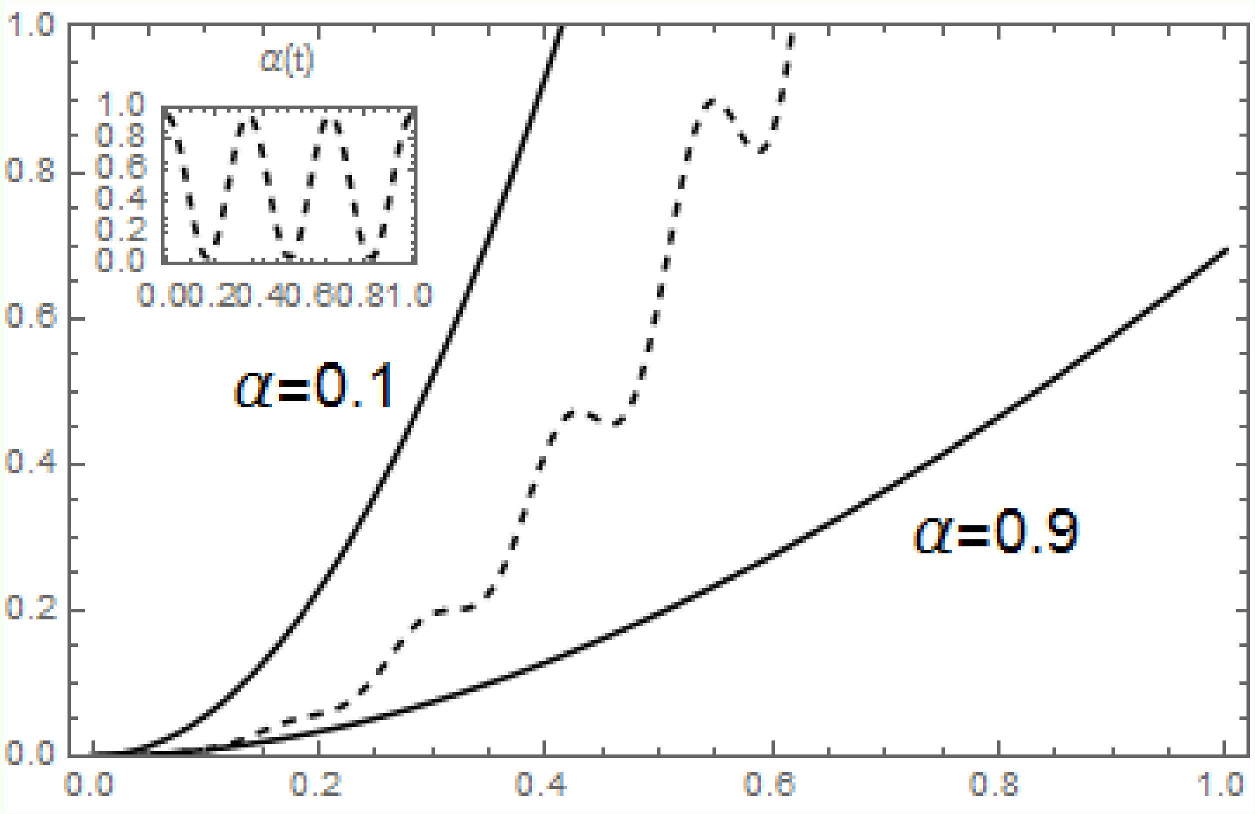
Solution for VODE equation with ∝ constant almost all interval $t \in (0, 1)$, except a narrow and sharp drop at $t = 0.5$. The two solid lines are exact solutions for these extreme values of ∝ and the dashed curve is the VODE numerical solution:



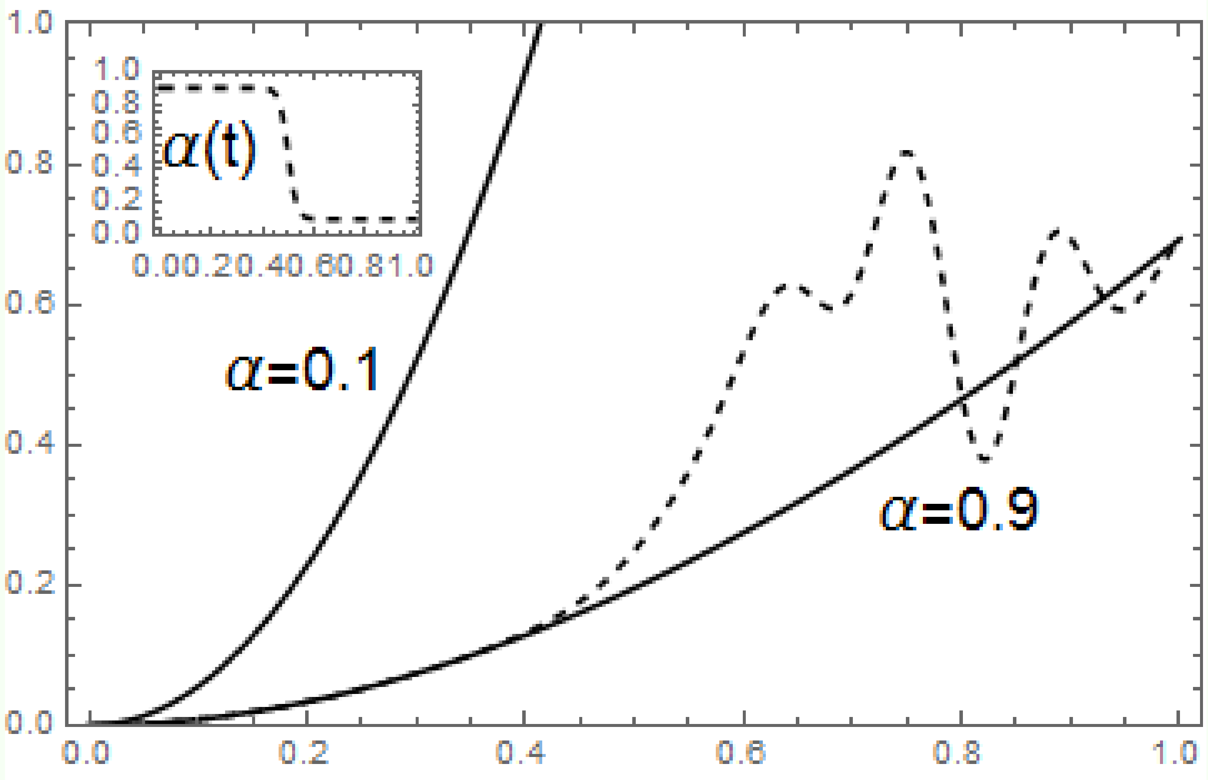
Solution for VODE equation with ∝ a step function at t = 0.5. The two solid lines are exact solutions for the extreme values of ∝ and the dashed curve is the VODE numerical solution:

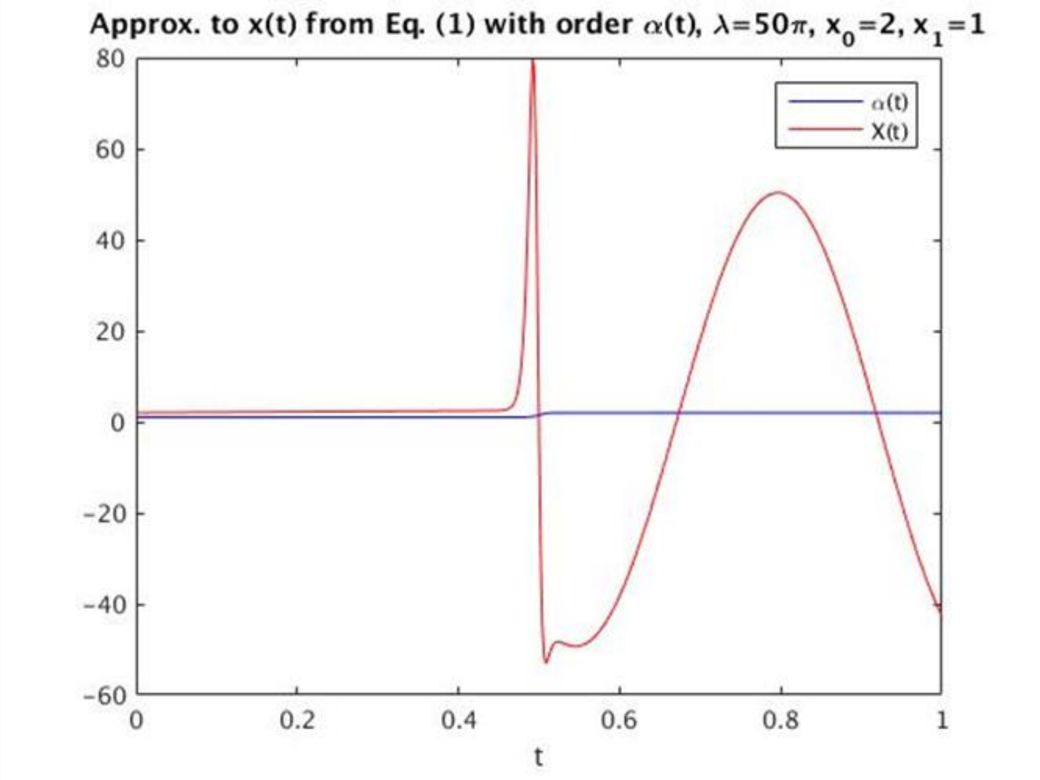


Solution for VODE equation with ∝ oscillating. The two solid lines are exact solutions for the extreme values of ∝ and the dashed curve is the VODE numerical solution:



Solution for VODE equation with ∝ a step function at t = 0.5. The two solid lines are exact solutions for the extreme values of ∝ and the dashed curve is the VODE numerical solution:





For the integer differential calculus, the tangent bundle TM over a manifold can be constructed for a given local differential structure with standard partial derivatives ∂_i.

Such an approach can be generalized to a fractional case when instead of ∂_i the differential structure is substituted, for instance, by the left Caputo derivatives:

$$\underline{D}_i^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{\partial^n f(x')}{\partial x'^n} \frac{dx'}{(x-x')^{\propto+1-n}}$$

with $\propto \in (n-1, n)$, for every local coordinate I on a local chart on M

Based on this we define the fractional tangent bundle $\underline{T}^{\propto} M$ for $\propto \in (0,1)$ where the components of the tangent vectors are Caputo derivatives.

With these vectors we can mimic the classic differential geometry and build local bases of vectors and the tangent fiber bundle.

We define fractional differential as dual local vector bases:

$$\boldsymbol{d}^{\alpha} = \sum_{j=1}^{n} \left(\boldsymbol{d}x^{j}\right)^{\alpha} \underline{\boldsymbol{D}}_{j}^{\alpha}$$

With this definition we build fractional differential 1-forms of component the functions F_j:

$$\omega^{\alpha} = \frac{1}{\Gamma(n-\alpha)} \sum_{j=1}^{n} d^{\alpha}\left(x^{j}\right)^{\alpha} F_{j}(x)$$

With the corresponding fractional exterior derivative we have a well defined fractional differential geometry structure on any smooth manifold M.

S. Vacaru, *J. Math. Phys.* **46** (2005) 042503; *Rep. Math. Phys.* **63** (2009) 95-110

D. Baleanu and S. Muslih, *Adv. Frac. Calc.* **2** (2007) 115-126

V. E. Tarasov, *Lett. Math. Phys.* **73** (2005) 49-58.

G. Perelman, arXiv: math. DG/ 03109

Next step is to apply fractional calculus towards a generalization of the Fundamental Theorem of Calculus.

With fractional differential geometry presented, we can introduce differential and integral vector operations, and the fractional Stokes' theorem.

By the duality endowed on M by cohomology and Stokes' formula we can introduce new definition for the boundary, i.e. the fractional boundary:

$$\int_{M} d^{\propto}\omega^{\propto} = \int_{B^{\propto}} \omega^{\propto}$$

Equation valid for all ω n-1 forms defined on all n-1 submanifolds of M, and to be solved for $B^{\propto}$ knowing that $B^{\propto=1} = \partial M$.

For example for n=1 and $\propto \in (0,1)$ the Riemann-Liouville fractional generalization of the Fundamental Theorem of Calculus does not work as expected:

$$ {}_{a}I_{b}^{\propto} \quad {}_{a}D_{x}^{\propto} f(x) = f(b) - \frac{(b-a)^{\propto}}{\Gamma(a)} \qquad {}_{a}I_{b}^{\propto -1} f(x) \neq f(b) - f(a) $$

In the present literature, for example:

V. E. Tarasov, *Ann. Physics* **323** (2008) 2756
J.T. Foley, A. J. Devaney, *Phys. Rev. B* **12** (1975) 3104
K. Cottrill-Shepherd, M. Naber, *J. Math. Phys.* **42** (2001) 2203, etc.

the authors used the Caputo derivative to avoid problems with generalization of this theorem.

Our approach is different. By keeping using the Riemann-Leibnitz fractional derivative we are able to present a well defined topological generalization of the Fundamental Theorem of Calculus .

In this case the fractional generalization of the Fundamental Theorem of Calculus predicts that the deformation, or "fractionation" of the boundary of the segment [a,b] is a {a, b} and a subset of this interval of non-zero Lebesgue measure.

Through ontogenesis and phylogenesis,
and considering the peripatetic, empiricist
"Nihil est in intellectu quod non prius fuerit in sensu"
we identify 2 or 3 ways of approaching math for a human
Geometry, Algebra, Statistics
created from human's most important senses: vision + touching, hearing, olfaction.

Therefore the question:

For a hypothetic family of human-smart cats or dogs, for whom the most important sense is smelling, what type of new mathematics could they create?

Space type	Range	Senses	Ontogenesis and phylogenesis of math	Physics-math modeling
Stereo	Long	Vision	2-dim geometry, topology	Waves (hyperbolic PDE)
		Hearing	Algebra	
	Local	Touch	3-dim geometry, topology	Thermo-elasticity (higher order PDE)
Stereo	Local	Olfaction	Statistics	Diffusion (parabolic PDE)
		Taste	Statistics	

Can this model explain:

1. Triggering events?
2. The rate of occurring of novelty
3. Dynamics
4. Change of topological and geometrical property of space?

From the first computations' results: YES

1. It can generate power laws for systems with moving boundary and there is one fit parameter only, the fractal dimension of the space.
2. It can mimic very well the rate of occurring of novelty for combinatorial models, urn models, criticality-phase transition models, patent law (highly nonlinear) model.
3. It has well defined dynamics
4. It involves the requested changes in topology and geometry of new adjacent potential boundaries.
5. Alternate models: Euler or Chern-Wyel characteristics: dimension of holes in a manifold is the number of times drawn.

Our preliminary simulations fit a large set of present and successful models:

Includes major hypotheses and features of the Kauffman's models for future, novelty, innovation, appears to be in agreement with the correlated novelties model (2014, 2017 – Tria, Loreto, Servedio, Strogatz,), **the combinatorial model for patents** (2015 - Youn, Strumsky, Bettencourt, Lobo), **population growth models** (2015 – Ribeiro & Ribeiro), **Heap's law model** (2005 – Leijenhorst, van der Weide), **neuronal time-scale models** (2015 – Zhigalov, Arnulfo, Nobili, Palva), **scientific fields topological transition results** (2015, 2016 – Bettencourt, Kaiser), **pace of global innovations in energy model** (2012 – Bettencourt, Trancik, Kaur), **scaling and collapsing sample space energy model** (2015 – Corominas-Murtra, Hanel, Thurner).

Also, there is a very good pure math backup from the work of: Jumarie, Caputo Samko, Kilbas, Hilfer, Tarasov, Cao, (fractional derivatives), G. Pereleman, Baleanu, Vacaru (fractional and Ricci flow), Whitney, Harrison (Stokes, Green), etc.

Of course there are still many issues and nuts and bolts to tight:

Ex. 1: The link between the early adoption of an innovation and its largescale spreading?

Ex. 2: Influence of laws and administrative constraints on creativity and production of novelty?

Ex. 3: Needs a better model-free explanation for the fractal dimension as free fitting parameter.

Etc. …

Looking very much forward for corrections, discussions, and collaboration,

Thank you for your patience!

APPENDICES

Assume that we have an ideal olfactory analyzer capable of labeling all possible stable gaseous chemical combinations, in all possible relative concentration by some tags chosen from a measurable space (N, μ). If the analyzer response is linear the subsets of N can be organized as an additive σ-algebra, and eventually as a measurable space.

Even in case of a detector with nonlinear response (like a real nose), having detection thresholds we can still construct a k-additive fuzzy measurable Sugeno-Grabisch space (N, F, μ).

However, these combinations of fragrances arrive at the nose through ontogenesis and phylogenesis experience and their effect into the brain will be different and nonlinear. So, everyone's life experience will map the nose's signals in specific way. It is like valuating each subset with a new measure, like a probability measure. In this way, the process of olfaction and fragrance memory could be organized as a mathematical statistics.

Stochastic Differential Equations (SDE) are related to Fokker-Planck equation (FPE) which is in a way similar to the Schrödinger equation. There is equivalence between FP equation and path integration theory through quantum mechanics and critical dynamics (FP equation can be transformed into the Schrödinger equation by rescaling a few variables). Every FP equation is equivalent to a path integral. When applied to particle position distributions SDE are equivalent to the convection–diffusion equations.

Conclusion:
Along with ontogenesis and phylogenesis human may have created three types of math in their brains, geometry, algebra and statistics, from their most important senses sight, hearing and the other local ones, respectively.

If a new family of very smart cats or dogs can develop, for whom the most important sense is smelling, what new type of mathematics could they have in their brains?

Animals can do a rough sort of math by summing sets of objects or sounds, or tell the difference between small and large amounts of numbers. [M. Tennesen, *Sci. Am.*, Sept. 2009]