The finite size analysis of chaotic systems

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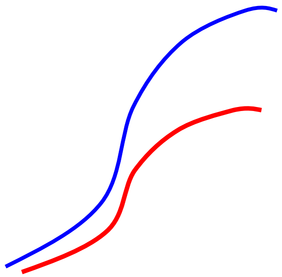
Outline

- Motivations
- Definition and computation of the FSLE
- Applications of FSLE to predictability problems
- Applications of FSLE to transport problems
- Conclusions

Starting question

What is the actual relevance of Lyapunov exponents to practical predictability problems?

First naive answer:



$$\delta(t) \approx \delta_0 e^{\lambda_1 t} \Longrightarrow \boxed{T(\delta_0, \Delta) \approx \frac{1}{\lambda_1} \ln\left(\frac{\Delta}{\delta_0}\right)}$$

True/reference trajectory $x(0)=x_0$

Perturbed/observed trajectory $x'(0)=x_0+\delta_0$

$$\delta(t) = |\boldsymbol{x}'(t) - \boldsymbol{x}(t)|$$

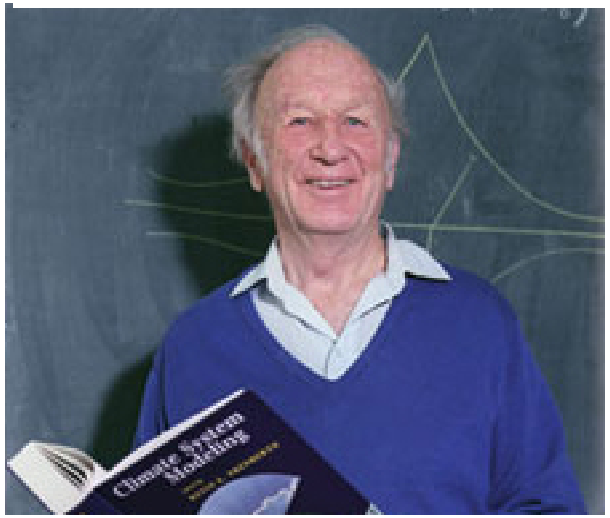
$$\lambda_1 = \lim_{t\to\infty} \lim_{||\delta\boldsymbol{x}(0)||\to 0} \frac{1}{t} \ln \frac{||\delta\boldsymbol{x}(t)||}{||\delta\boldsymbol{x}(0)||}$$

Let us identify the main aspects of the problem

Predictability – a problem partly solved

Edward N. Lorenz

In predictability vol I
ECWF Seminar, Reading UK 1996



Lorenz 1996 models

$$dX_k/dt = -X_{k-2}X_{k-1} + X_{k-1}X_{k+1} - X_k + F,$$

or

$$dX_k/dt = -X_{k-1}(X_{k-2} - X_{k+1}) - X_k - (hc/b)\sum_{j=1}^{J} Y_{j,k},$$

$$dY_{j,k}/dt = -cbY_{j+1,k}(Y_{j+2,k} - Y_{j-1,k}) - cY_{j,k} + (hc/b)X_k$$

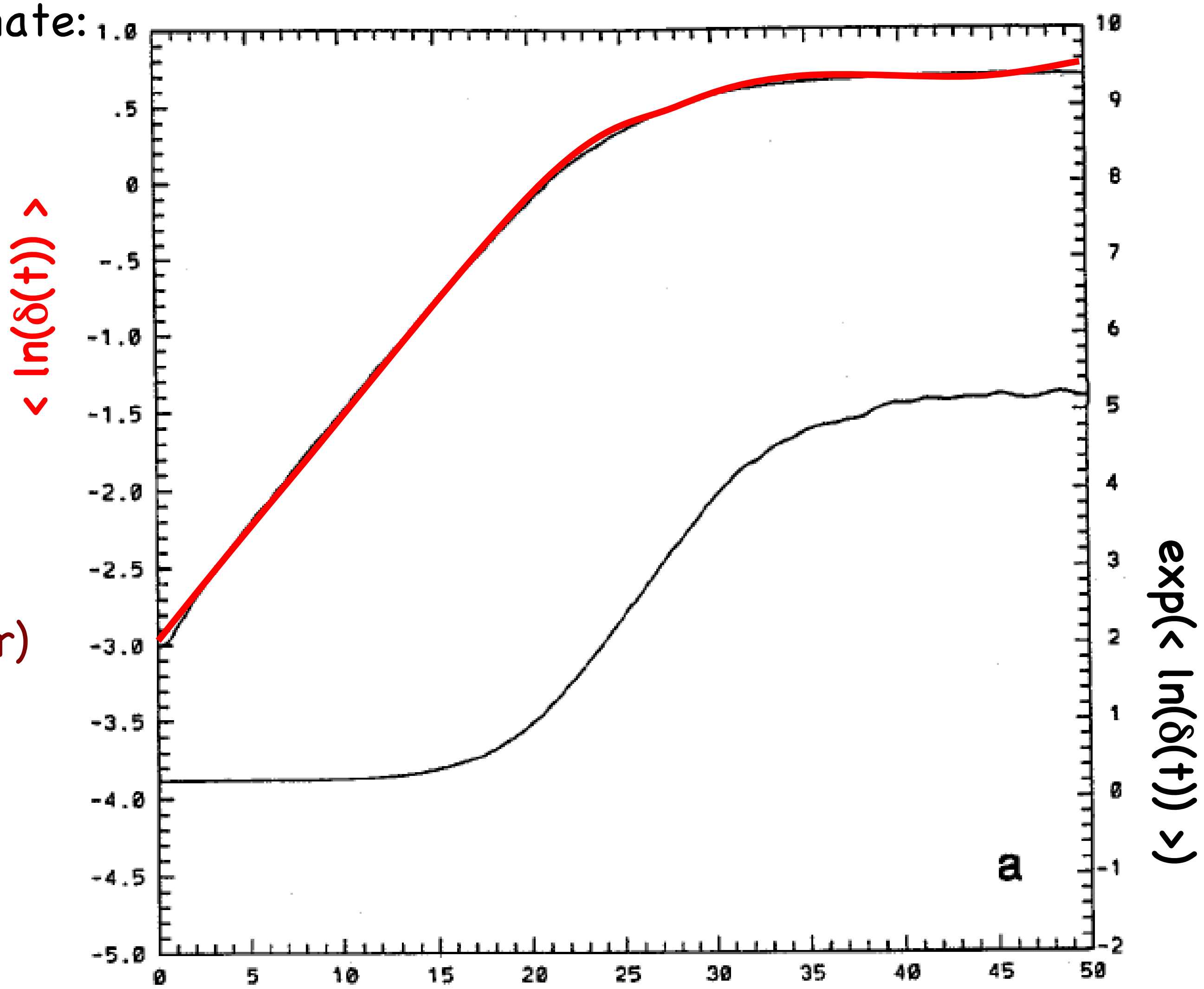
With the aid of some simple models, we describe situations where errors behave as would be expected from a knowledge of λ_1, and other situations, particularly in the earliest and latest stages of growth, where their behaviour is systematically different. Slow growth in the latest stages may be especially relevant to the long-range predictability of the atmosphere. We identify the predictability of long-term climate variations, other than those that are externally forced, as a problem not yet solved.

Different regimes of error growth

$$dX_k/dt = -X_{k-2}X_{k-1} + X_{k-1}X_{k+1} - X_k + F,$$

What we want to estimate:
e.g. Error doubling time
T(δ,Δ) with Δ=2δ

What we have:
1. Lyapunov exponent
2. Saturation level
   (size of the attractor)
3. Initial error δ



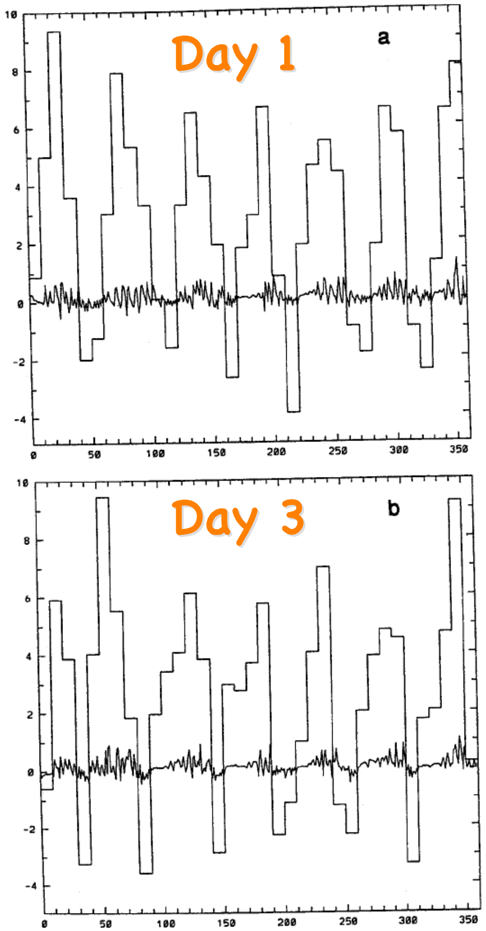
LE in the presence of different time scales

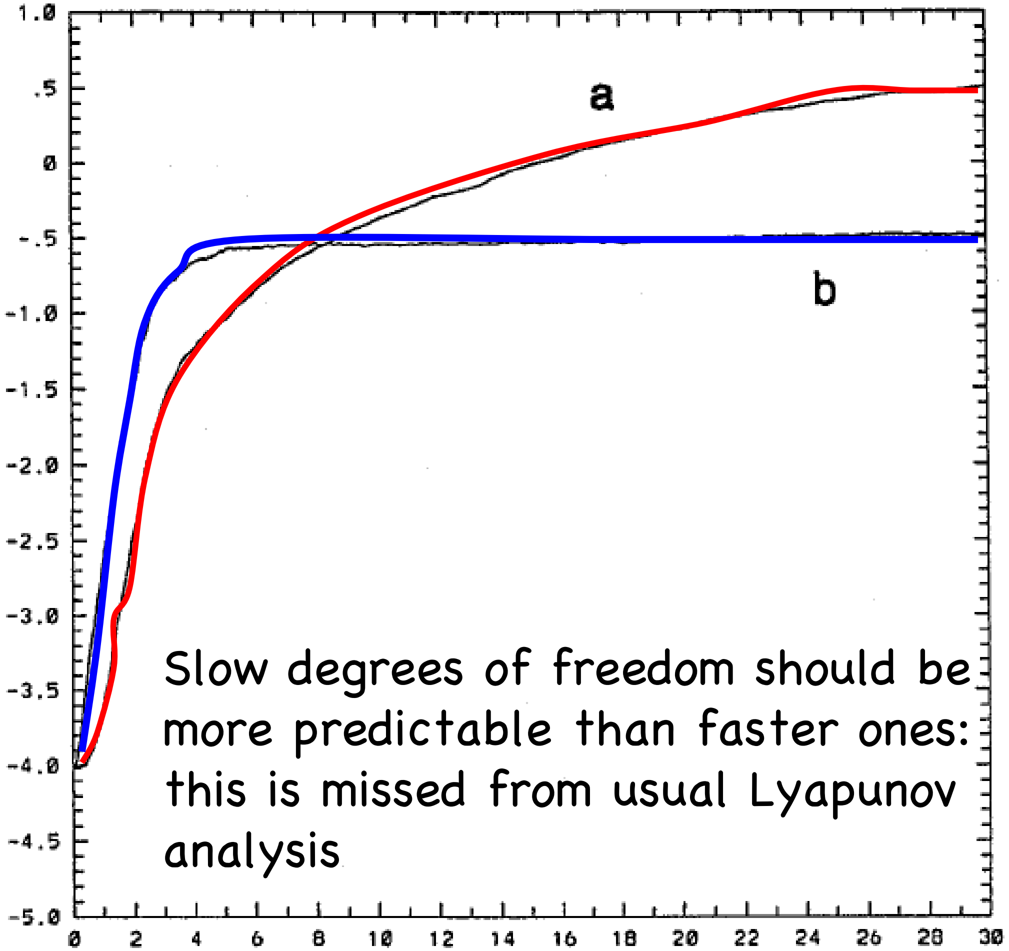
$$dX_k/dt = -X_{k-1}(X_{k-2} - X_{k+1}) - X_k - (hc/b)\sum_{j=1}^{J} Y_{j,k},$$

Slow (synoptic scales)

$$dY_{j,k}/dt = -cbY_{j+1,k}(Y_{j+2,k} - Y_{j-1,k}) - cY_{j,k} + (hc/b)X_k.$$

Fast (convective scales)





Slow degrees of freedom should be more predictable than faster ones: this is missed from usual Lyapunov analysis

Main problems with usual LE

- It is an asymptotic quantity while in practice we are typically far from any asymptotics (flucutations of the growth rate can be important)

Despite the agreement between the error growth in the simple model, and even in some global circulation models, with simple first estimates, reliance on the leading Lyapunov exponent, in most realistic situations, proves to be a considerable over-simplification. By and large this is so because λ_1 is defined as the long-term average growth rate of a very small error. Often we are not primarily concerned with averages, and, even when we are, we may be more interested in shorter-term behaviour. Also, in practical situations the initial error is often not small.

- In particular, both the amplitude of the initial error or of the chosen tolerance matters!

When the initial error is not particularly small, as is often the case in operational weather forecasting, λ_1 may play a still smaller role. The situation is illustrated by

- In the presence of many characteristic times, the LE only accounts for the fastest one, while often the slowest ones are of interest

The relevance of the Lyapunov exponent is even less certain in systems, such as more realistic atmospheric models or the atmosphere itself, where different features possess different characteristic time scales. In fact, it is not at all obvious what the leading exponent for the atmosphere may be, or what the corresponding vector may look like. To gain some insight, imagine a relatively realistic model that resolves

The finite-size Lyapunov exponent

We want an indicator able to characterize the error growth rate at changing the scale (of the initial error δ and/or of our tolerance Δ)

$$\delta(t) \approx \delta_0 e^{\lambda_1 t} \Longrightarrow T(\delta_0, \Delta) \approx \frac{1}{\lambda_1} \ln\left(\frac{\Delta}{\delta_0}\right)$$

?

$$\text{“} \lambda(\delta, \Delta) = \frac{1}{T(\delta, \Delta)} \ln\left(\frac{\Delta}{\delta}\right) \text{”}$$

Aurell, Boffetta, Crisanti, Paladin & AV PRL 1996 J.Phys. A 1997

Mathematical difficulties

When considering non-infinitesimal uncertainties we face the problem of dependence on the norm, furthermore we are forced to work in preasymptotic situations where establishing rigorous mathematical results is rather difficult.

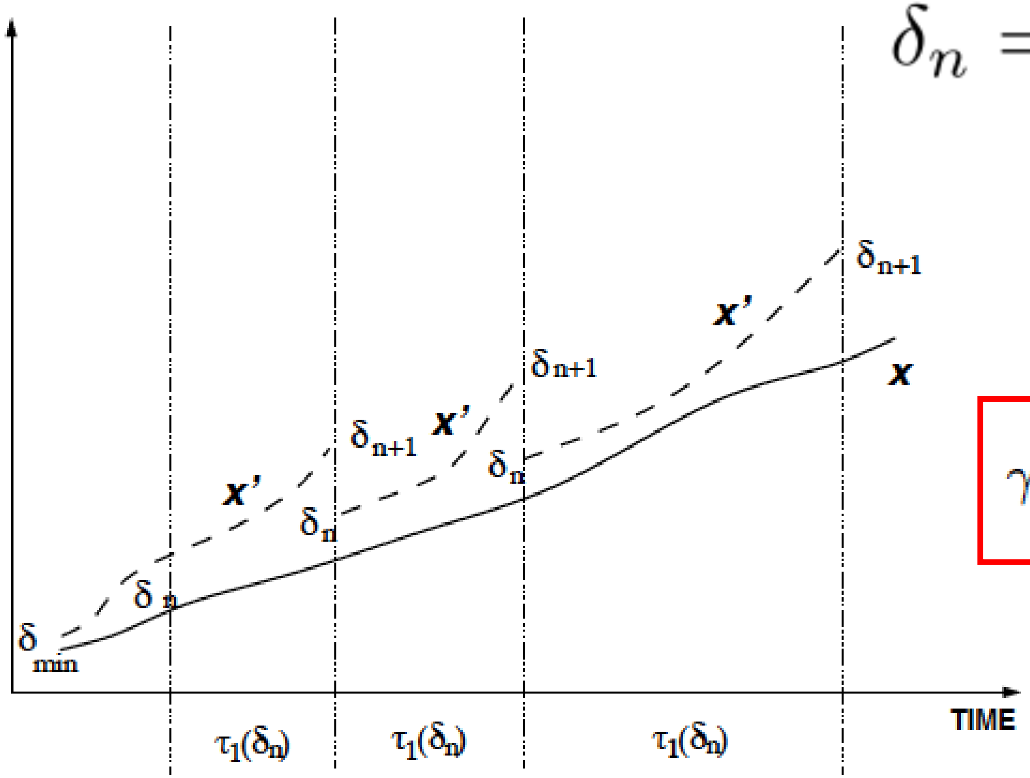
Physical solution

Abandoning the request of mathematical rigor, we can attempt an operative (algorithmic) definition of an appropriate indicator such to ensure that the usual LE is obtained in the proper limits, the idea is to use norms suggested by physical intuition

$$\delta = ||x - x'||$$

Note: another possible quantity is the ε-entropy (Kolmogorov et al 1956), but it is practically uncomputable & depends on the norm anyway

An algorithmic definition (I)



$$\delta_n = \delta_0 \varrho^n$$ n=1,...,N thresholds

$\tau_i(\delta_n)$ first time for the error to grow from δ_n to δ_{n+1}

i=1,...,$\mathcal{N}_d$ "doubling" time experiments

$$\gamma_i(\delta_n) = \frac{1}{\tau_i(\delta_n)} \ln \varrho$$

Growth rate at scale δ_n

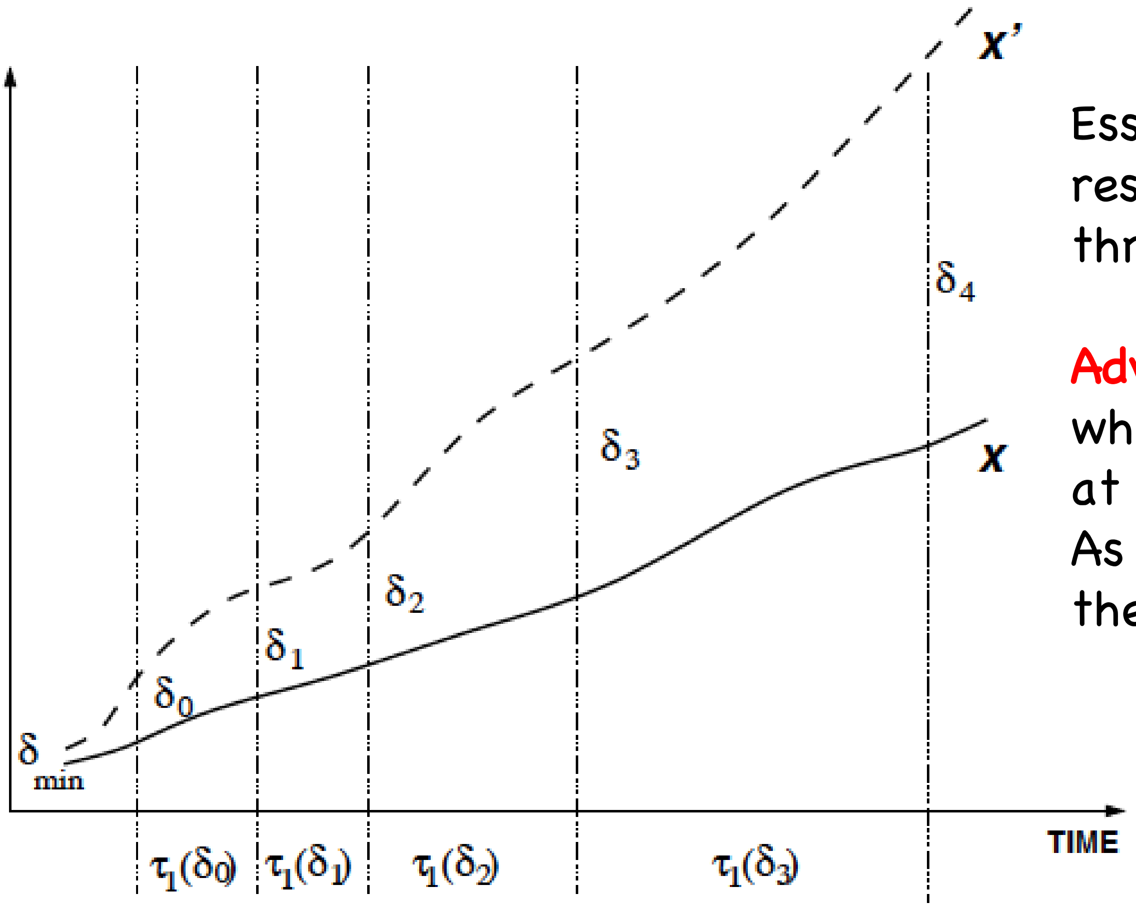
$$T = \sum \tau_i \quad \langle \tau(\delta_n) \rangle_d = \frac{\sum \tau_i}{\mathcal{N}_d}$$

FSLE

$$\lambda(\delta_n) = \langle \gamma(\delta_n) \rangle_t = \frac{1}{T} \int_0^T \mathrm{d}t\, \gamma = \frac{\sum_i \gamma_i \tau_i}{\sum_i \tau_i} = \frac{\ln \varrho}{\langle \tau(\delta_n) \rangle_d}$$

For δ_n->0 Benettin et al. Algorithm ==> $\lim_{\delta \to 0} \lambda(\delta) = \lambda_1$

An algorithmic definition (II)



Essentially the same idea but rescaling only when the last threshold is reached

Advantage:
when performing the rescaling at non-infinitesimal perturbations
As in (I) we may exit from the attractor

Typically the two methods provide very close results

Application to predictability problems

- First Kind: when one assume to have a perfect model but an error on the knowledge of the initial condition

- Second Kind: when the exact dynamics is uncertain and the model is thus known with some error (parameters, unresolved degrees of freedom etc...)

FLSE for a system with 2 times scales

Coupled Lorenz '69 models with different time scales

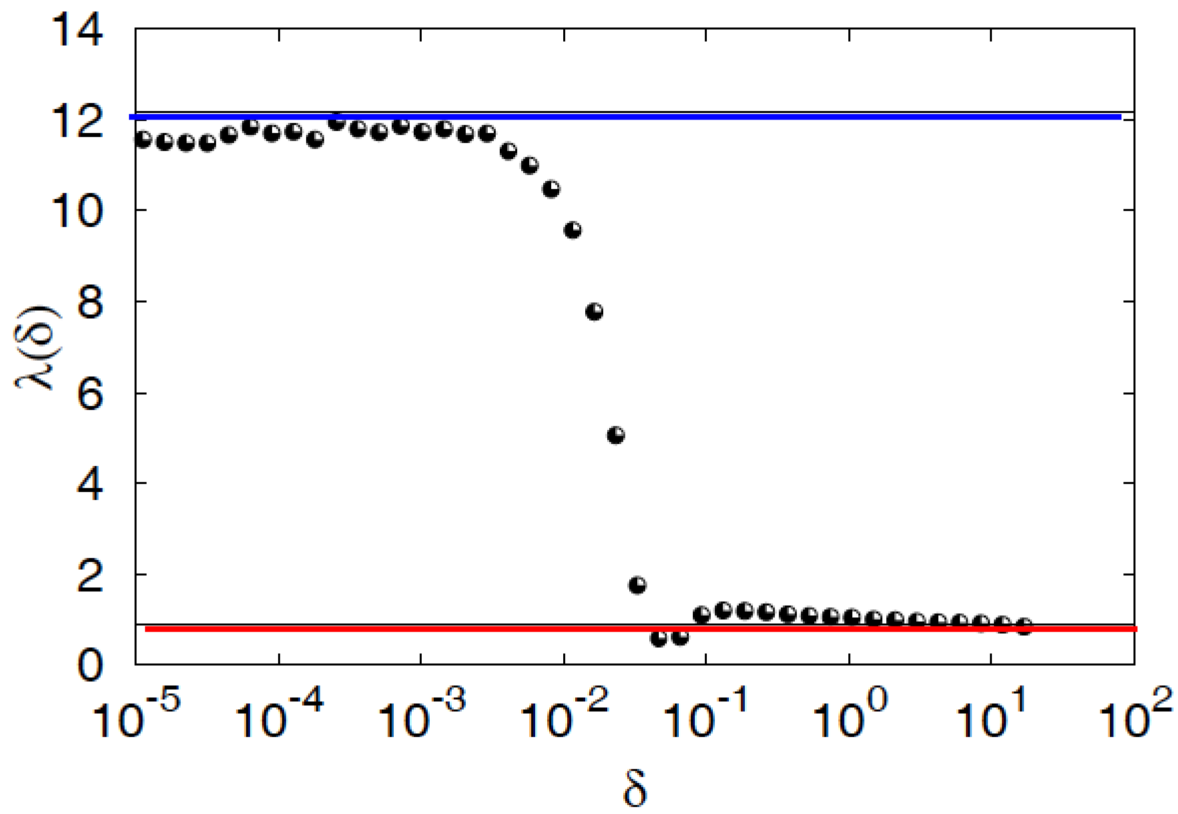
$$\frac{dx_1^{(s)}}{dt} = \sigma(x_2^{(s)} - x_1^{(s)})$$
$$\frac{dx_2^{(s)}}{dt} = (-x_1^{(s)}x_3^{(s)} + r_s x_1^{(s)} - x_2^{(s)}) - \epsilon_s x_1^{(f)} x_2^{(f)}$$
$$\frac{dx_3^{(s)}}{dt} = x_1^{(s)}x_2^{(s)} - b x_3^{(s)}$$

slow

$$\frac{dx_1^{(f)}}{dt} = c\,\sigma(x_2^{(f)} - x_1^{(f)})$$
$$\frac{dx_2^{(f)}}{dt} = c\,(-x_1^{(f)}x_3^{(f)} + r_f x_1^{(f)} - x_2^{(f)}) + \epsilon_f x_1^{(f)} x_2^{(s)}$$
$$\frac{dx_3^{(f)}}{dt} = c\,(x_1^{(f)}x_2^{(f)} - b x_3^{(f)}),$$

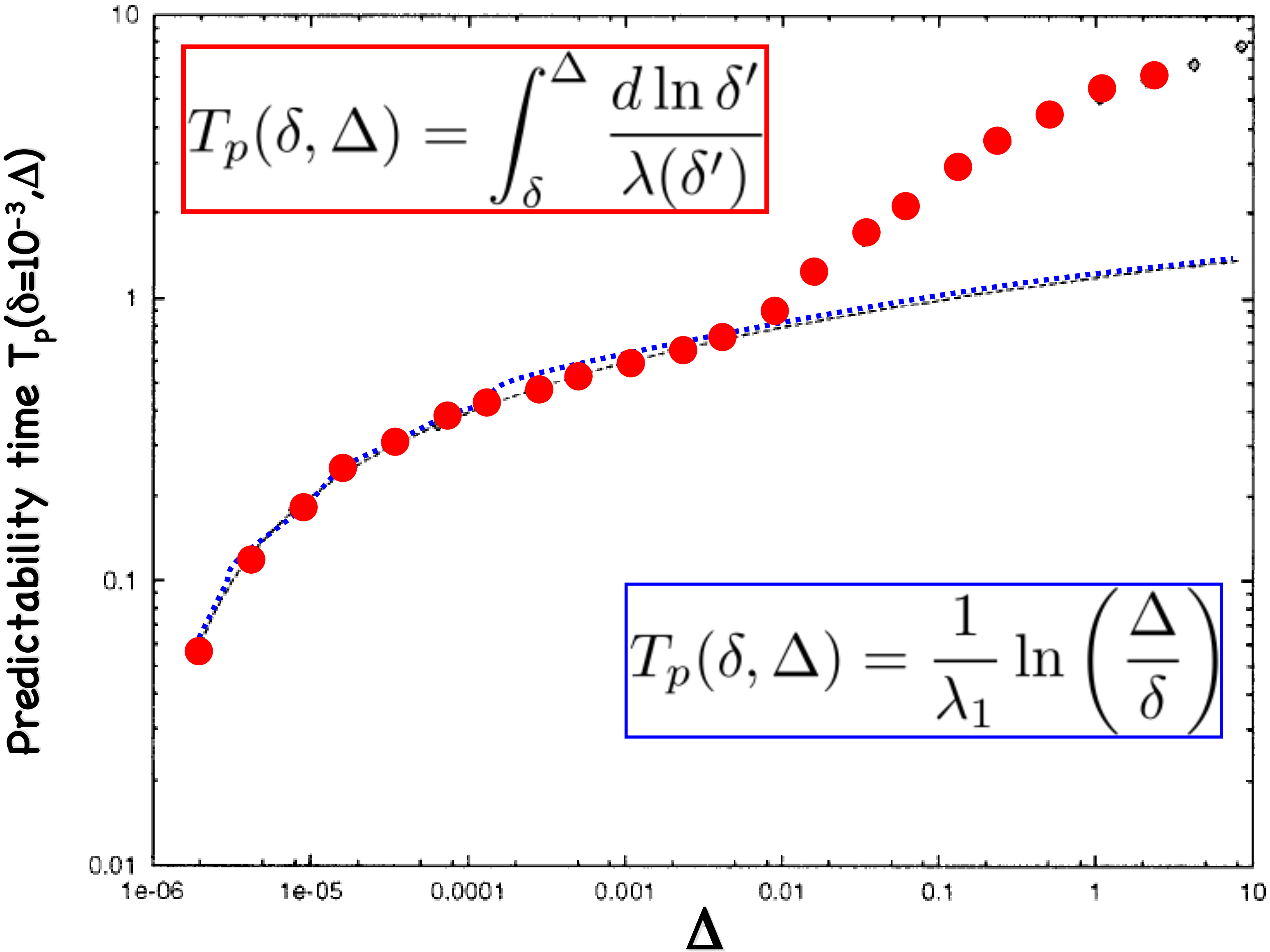
fast

$$\delta = ||\boldsymbol{x}^{(s)} - \boldsymbol{x}^{(s)'}||$$



Boffetta, Giuliani, Paladin & AV J. Atm. Sci. 1998

Predictability in a system with 2 times scales



Example: parametrization of fast time scales

True model

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

x=slow
y=fast

Approximated model
(parametrization of small/fast scales)

$$\frac{dx}{dt} = f_M(x, y(x))$$

For instance

Lorenz 1996
"true" ->

$$\frac{dx_k}{dt} = -x_{k-1}(x_{k-2} - x_{k+1}) - \nu x_k + F - \sum_{j=1}^{J} y_{j,k}$$
$$\frac{dy_{j,k}}{dt} = -cby_{j+1,k}(y_{j+2,k} - y_{j-1,k}) - c\nu y_{j,k} + x_k$$

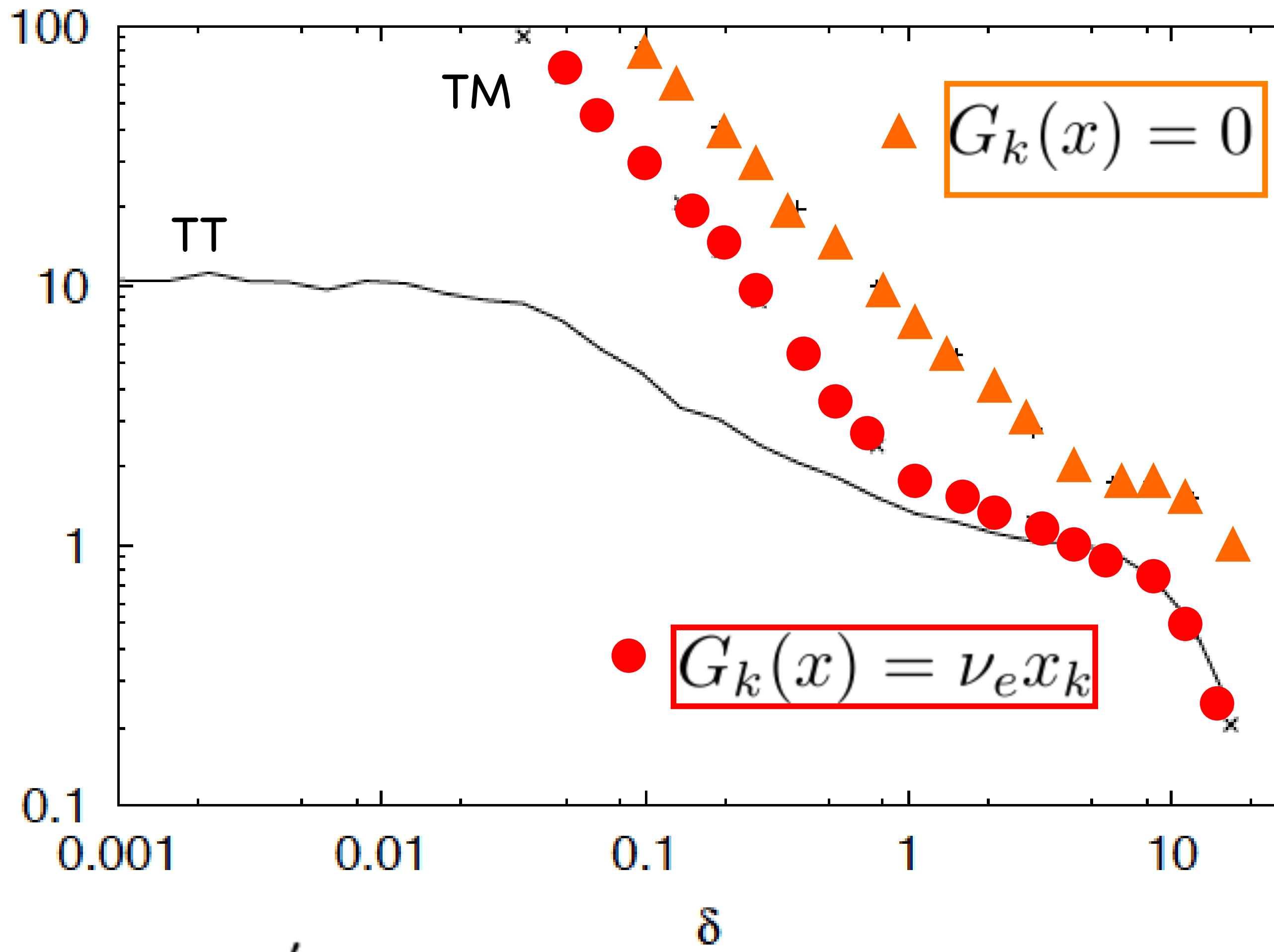
"model"->

$$\frac{dx_k}{dt} = -x_{k-1}(x_{k-2} - x_{k+1}) - \nu x_k + F - G_k(x)$$

Boffetta, Cencini, Celani, Lacorata & AV J. Phys. A 2000

FSLE and test of quality of the parametrization

T=true
M=model

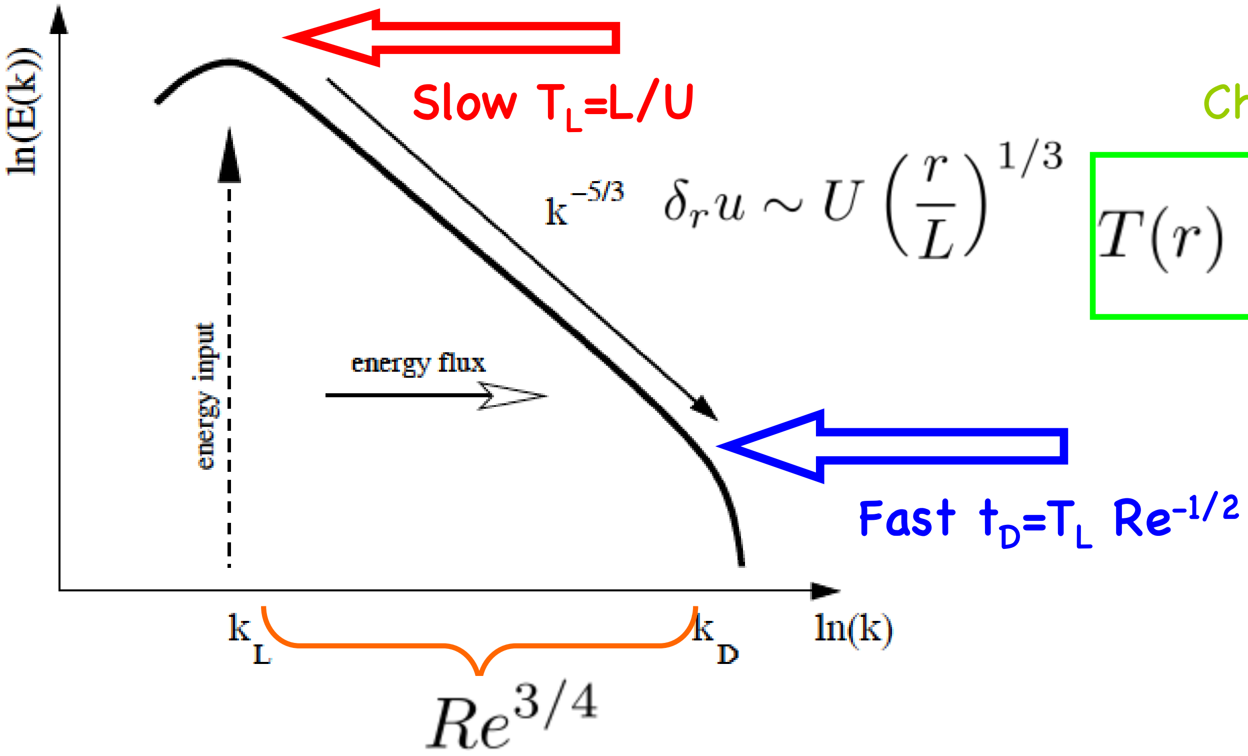


$$\delta = ||\boldsymbol{x}_T - \boldsymbol{x}'_{T,M}||$$

Predictability & FSLE in turbulence

Turbulence is characterized by the presence of many characteristic scales and time scales

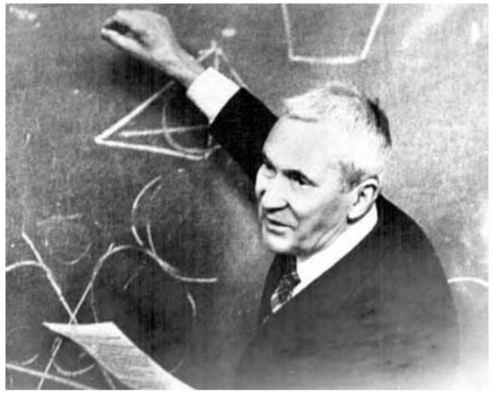
Basic ideas of classical Kolmogorov (1941) phenomenology



$$\delta_r u \sim U\left(\frac{r}{L}\right)^{1/3}$$

Characteristic times

$$T(r) \approx \frac{r}{\delta_r u} \approx T_L\left(\frac{r}{L}\right)^{2/3}$$



Predictability & FSLE in turbulence

Classical theory of predictability (Lorenz 1969, Leith 1971, Leith & Kraichnan 1972)

Basic Idea: Given a perturbation on the velocity field at scale r/2 scale r will be completely uncertain after a time $T(r) \approx T_L (r/L)^{2/3}$

Therefore an uncertainty at the smallest active scale r_D will reach L in a time

$$T_p(r_D, L) \sim T(r_D) + T(2r_D) + \ldots + T(L/2) + T(L) \sim T_L = L/U$$

Standard LE

$$\lambda_1 \approx \frac{1}{T(r_D)} \approx \frac{Re^{1/2}}{T_L} \quad \Longrightarrow \quad T_p \sim \frac{1}{\lambda_1} \sim \frac{T_L}{Re^{1/2}}$$

FSLE

$$T(r) = \frac{r}{\delta_r u} \Longrightarrow T(\delta u) = T_L \left(\frac{\delta u}{U}\right)^2 \quad \Longrightarrow \quad \lambda(\delta u) \sim \begin{cases} \lambda_1 & \delta u < \delta_{r_D} u \\ (\delta u)^{-2} & \delta_{r_D} u < \delta u < U \end{cases}$$

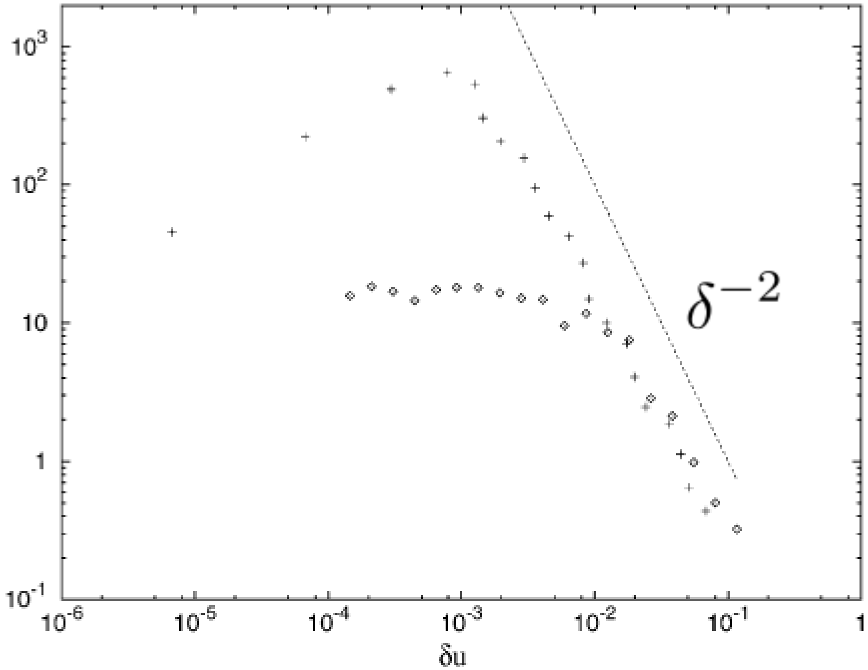
This result holds also considering Corrections to K41

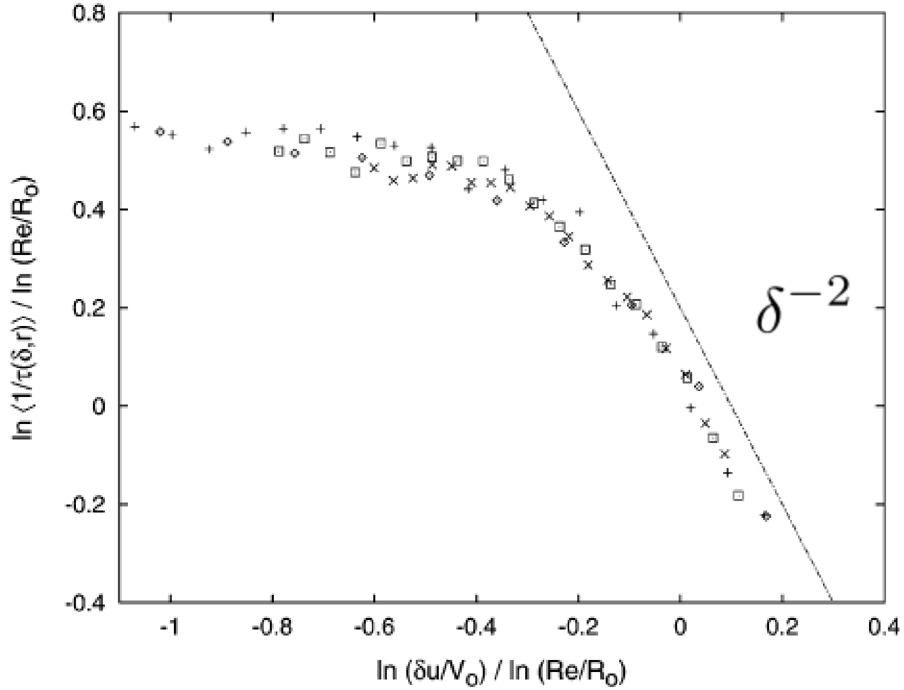
FSLE in shell models for 3d turbulence

$$\partial_t v + v \cdot \nabla v = -\frac{1}{\rho}\nabla p + \nu \Delta v + f$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu k_n^2\right) u_n = i\left[k_n u_{n+1}^* u_{n+2}^* - \delta k_{n-1} u_{n-1}^* u_{n+1}^* - (1-\delta) k_{n-2} u_{n-2}^* u_{n-1}^*\right] + f_n$$

$$\delta = \left\|u - u'\right\|$$



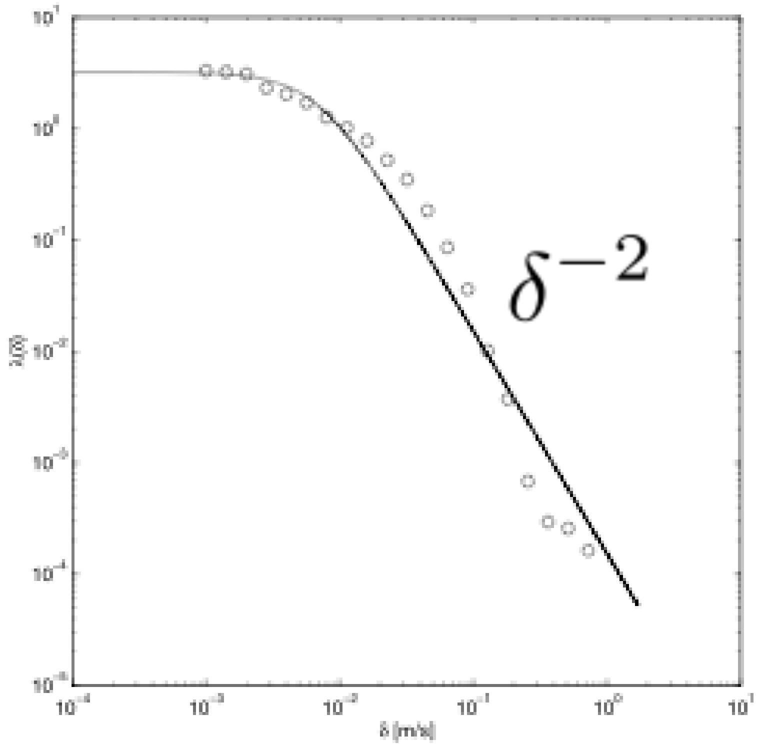


Aurell, Boffetta, Crisanti, Paladin & AV. PRL 1996 J.Phys. A 1997

FSLE in Atmospheric Boundary layer

Predictability of atmospheric boundary-layer flows as a function of scale GRL 2002

S. Basu, E. Foufoula-Georgiou, and F. Porté-Agel



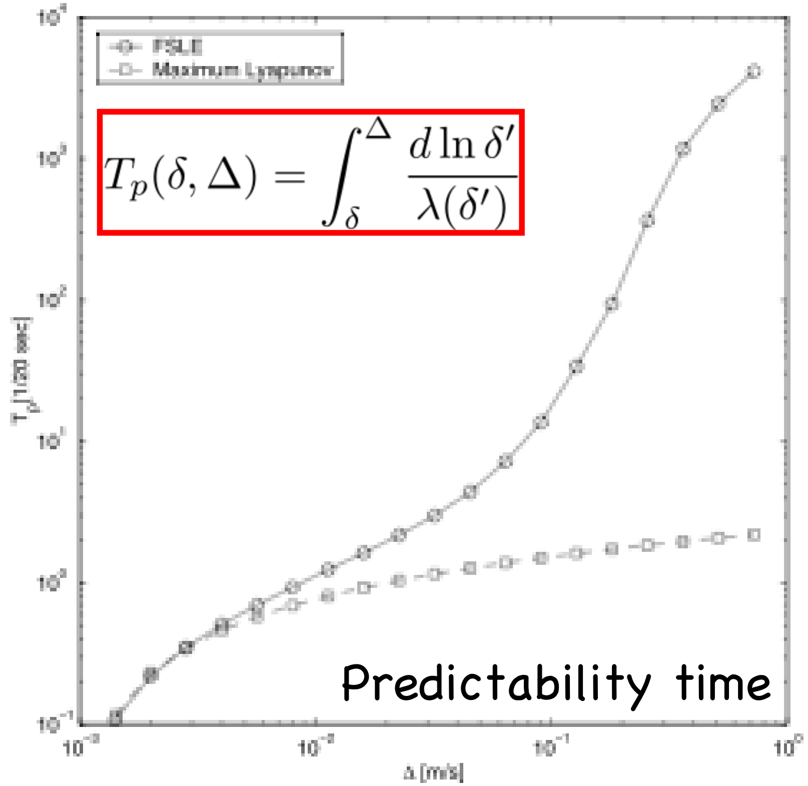


Figure. 5. Predictability time T_p based on FSLE and Maximum Lyapunov Exponent for the series u_A (initial error of 10^{-3} m/s). Note that predictability inferred by assuming infinitesimal perturbations (dashed line/squares) is much smaller than that inferred by finite-size perturbations (solid line/circles).

Applications of FSLE to transport problems

The FSLE quantifies the growth rate between separating trajectories of a dynamical system at changing the scale of separation, it is thus best suited to charcterize trajectories separation in situations in which different physical mechanisms are acting at different length scales

E.g. relative dispersion of tracers in fluid flows

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{v}(t) = \boldsymbol{u}(\boldsymbol{X}(t), t) \qquad \frac{d\boldsymbol{R}}{dt} = \boldsymbol{u}(\boldsymbol{X}_1(t), t) - \boldsymbol{u}(\boldsymbol{X}_2(t), t) = \delta_R \boldsymbol{u}$$

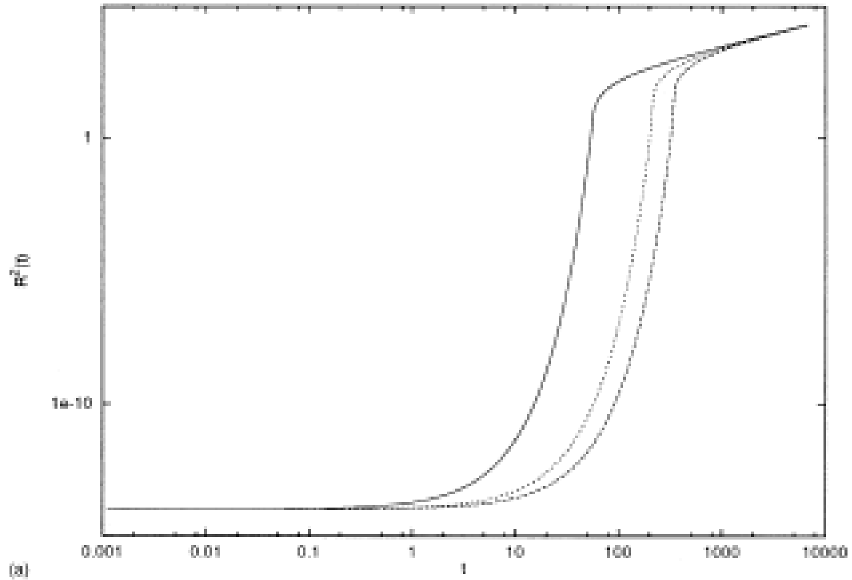
FSLE is particularly useful when coping with situations in which asymptotic regimes cannot be established due to the presence of boundaries

or when $\delta_R u$ depends on R inducing different regimes for the evolution of R(t), e.g. turbulence

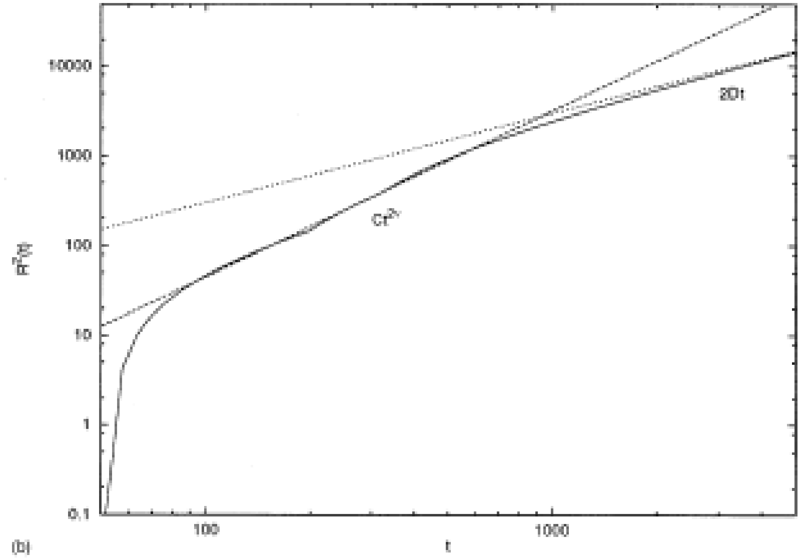
$$\delta_R \boldsymbol{u} \sim \begin{cases} R & R < r_D \\ R^{1/3} & r_D < R < L \\ U & r > L \end{cases}$$

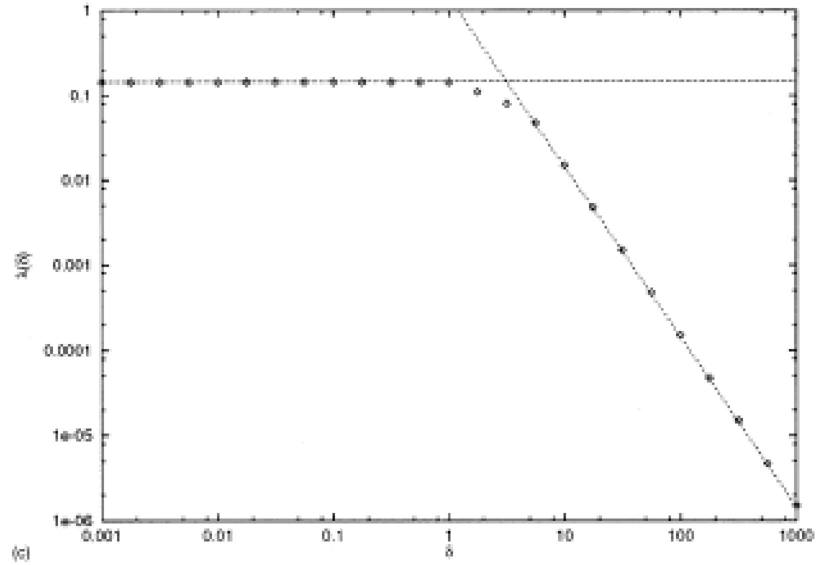
Main advantage of FSLE fixing the scale may eliminate
Spurious effects appearing with the standard fixed time analysis

Fixed scale vs fixed time: a simple example



$$R^2(t) = \begin{cases} \delta_0^2 \exp(2\gamma t) & R < 1 \\ 2D(t - t^*) & R > 1 \end{cases}$$

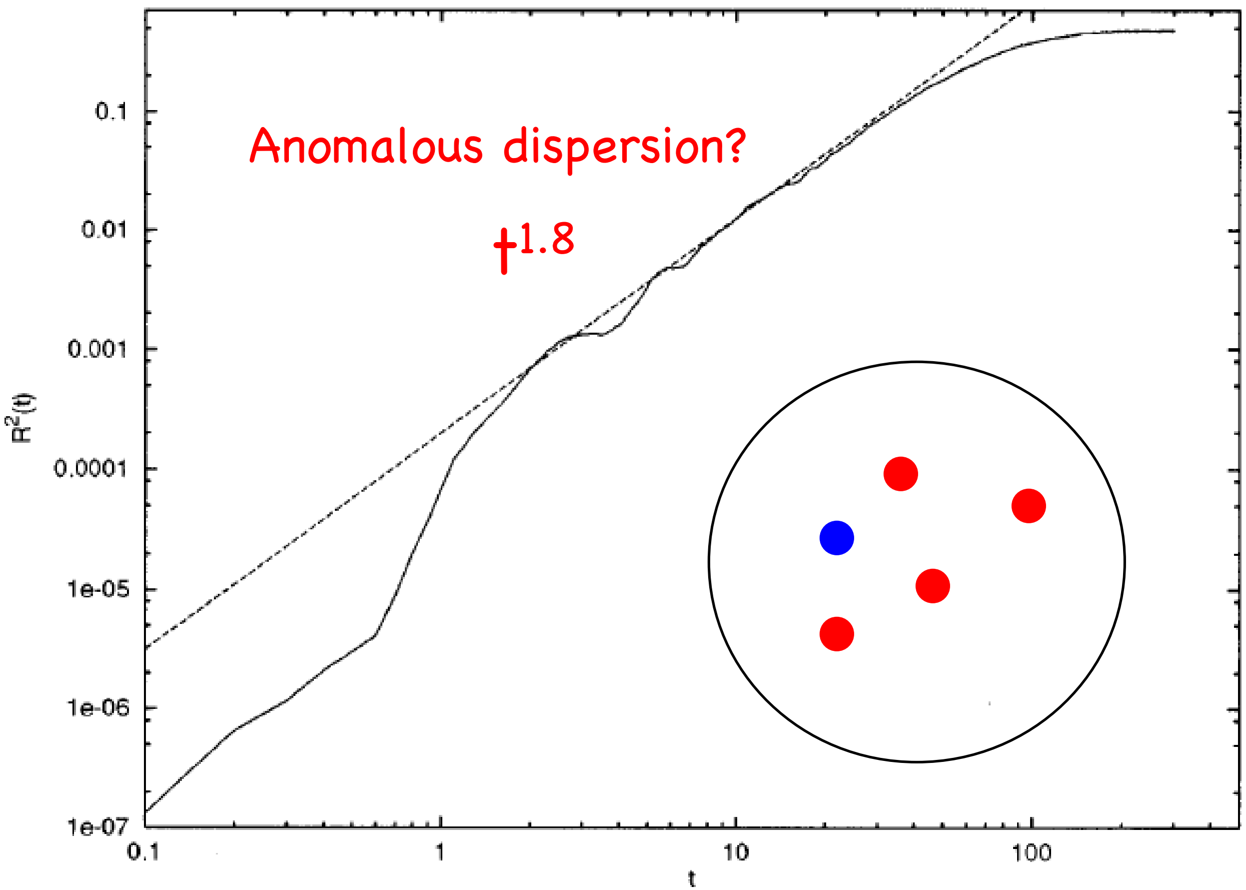




Dispersion in non-asymptotic situations

Dispersions of tracers
In a disk with 4 point vortices

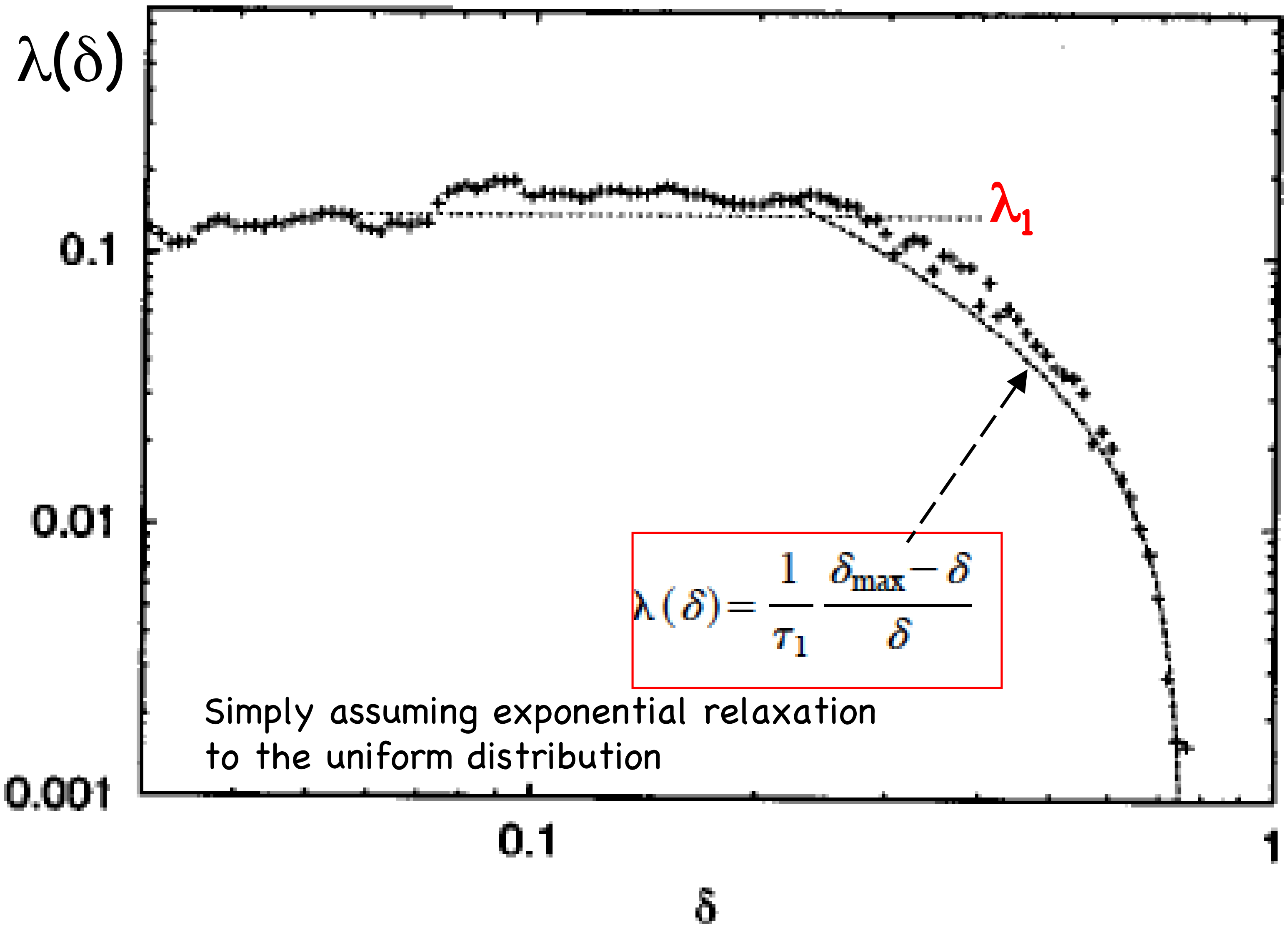
$$\dot{x}_i = \frac{1}{\Gamma_i} \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{1}{\Gamma_i} \frac{\partial H}{\partial x_i}$$



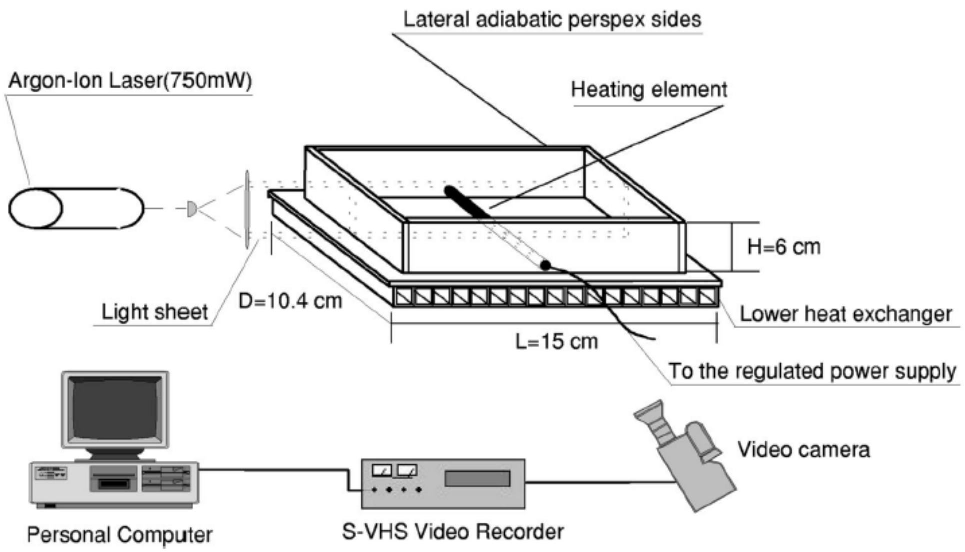
Due to the finite domain no time asyptotics and thus difficulty of interpretation

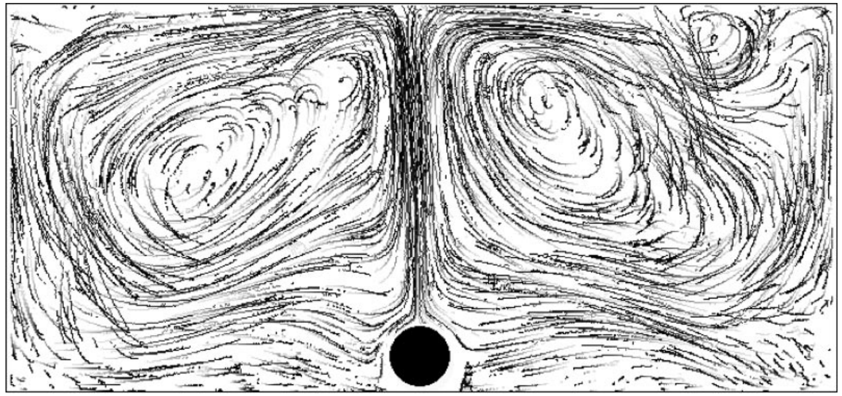
Artale, Boffetta, Cencini, Celani & AV Phys. Fluids 1997

Dispersion in non-asymptotic situations

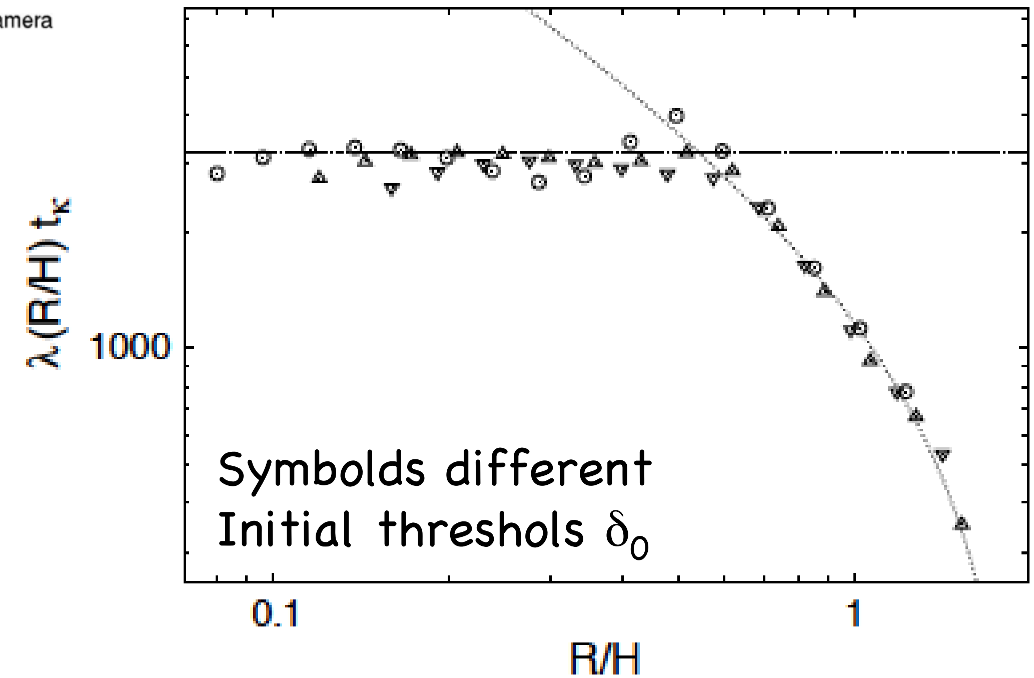


Easy to compute in experiments





Experiment (PIV) & data analysis



Boffetta, Cencini, Espa & Querzoli
EPL 1999 & Phys. Fluids 2000

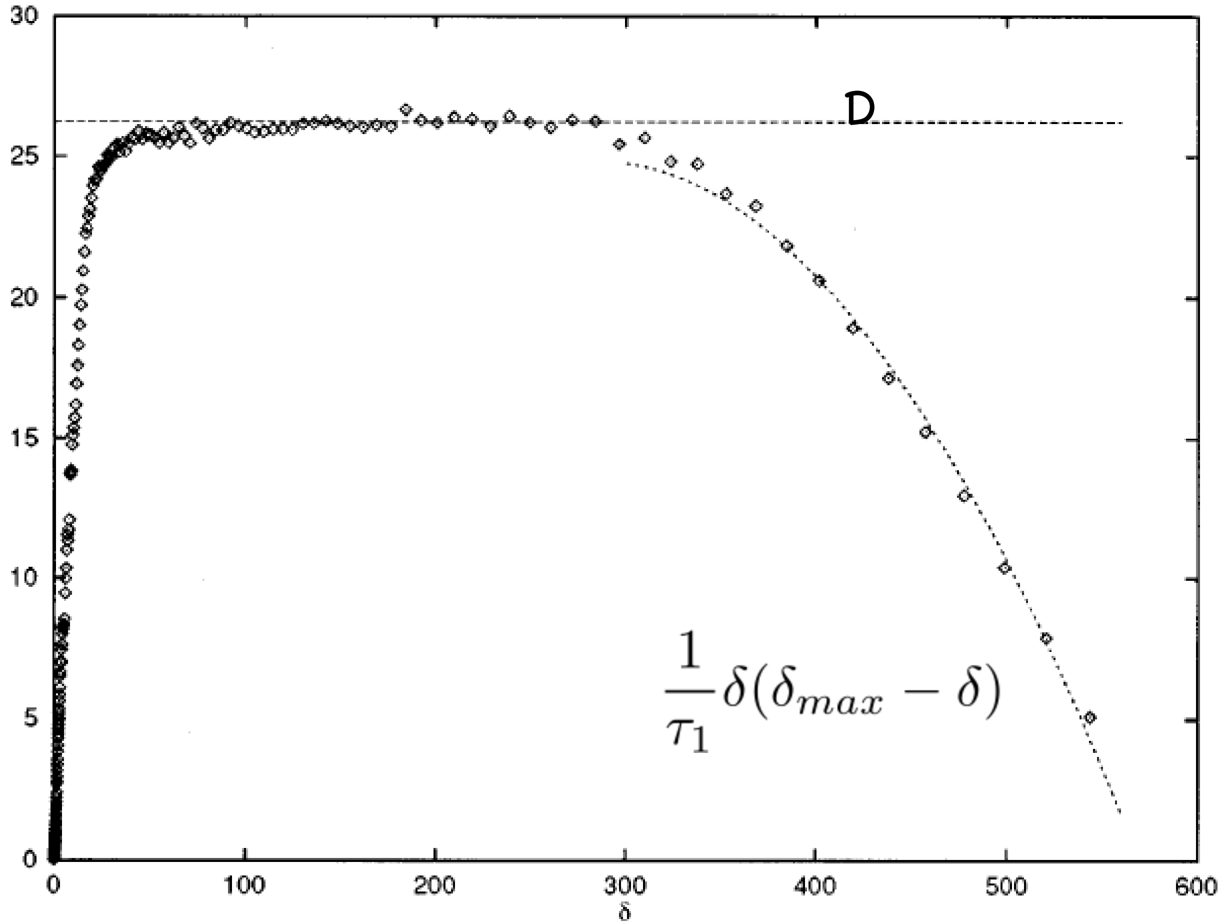
Scale dependent diffusion coefficient

$$x_{n+1} = x_n + K f(x_{n+1}) \sin y_n$$

$$y_{n+1} = y_n + x_{n+1} - K f'(x_{n+1}) \cos y_n \quad \text{mod } 2\pi$$

$$f(x) = \begin{cases} 1, & |x| < \ell \\ \dfrac{L - |x|}{L - \ell}, & \ell < |x| < L \end{cases}$$

$$D(\delta) = \delta^2 \lambda(\delta)$$

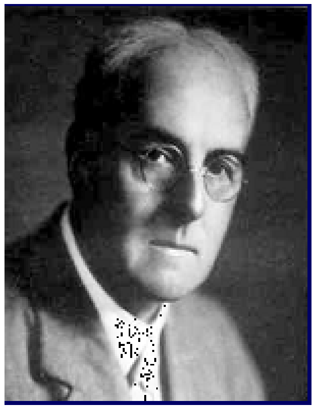


Artale, Boffetta, Cencini, Celani & AV Phys. Fluids 1997

Relative dispersion in turbulence

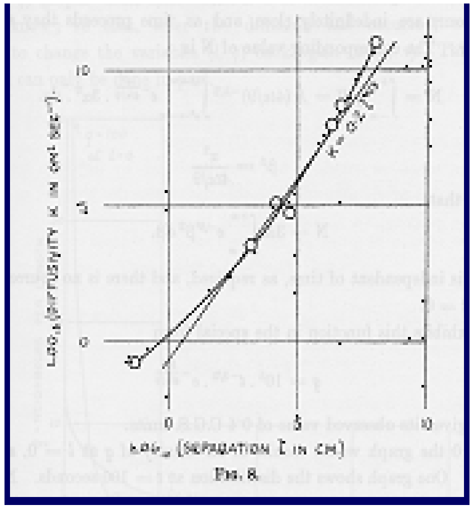
$$\frac{d\boldsymbol{R}}{dt} = \boldsymbol{u}(\boldsymbol{X}_1(t), t) - \boldsymbol{u}(\boldsymbol{X}_2(t), t) = \delta_R \boldsymbol{u}$$

Chaotic dispersion	Richardson Dispersion	Standard Dispersion
$\delta_R u \propto R$	$\delta_R u \propto R^{1/3}$	$\delta_R u \sim U$
$R(t) \sim R(0)\exp(\lambda_1 t)$	$R^2(t) \sim t^3$	$R^2(t) \sim 2D_{eff}t$
$\lambda(R) \approx \lambda_1$	$\lambda(R) \approx \delta^{-2/3}$	$\lambda(R) \approx D_{eff}\delta^{-2}$



Richardson 1926

$$D(R) \approx R^2\lambda(R) \sim R^{4/3}$$



Fixed scale vs Fixed time in turbulence

In principle one may think that the FSLE $\lambda(\delta)$ is the same as

$$\tilde{\lambda}(\delta) = \frac{1}{2\left\langle R^2(t)\right\rangle} \frac{\mathrm{d}\left\langle R^2(t)\right\rangle}{\mathrm{d}t}\Bigg|_{\langle R^2\rangle=\delta^2} \qquad \tilde{\lambda}(\delta) = \frac{\mathrm{d}\left\langle \ln R(t)\right\rangle}{\mathrm{d}t}\Bigg|_{\langle \ln R(t)\rangle=\ln\delta}$$

BUT is not:
the main point is that $\lambda(\delta)$ depends only on the scale δ
while $\tilde{\lambda}(\delta)$ will depend also on R(0) this is particularly relevant to turbulence
where looking at fixed times R(t) is strongly ifluenced by R(0)

Relative dispersion in turbulence

1024^3 DNS

Usual Fixed time analysis

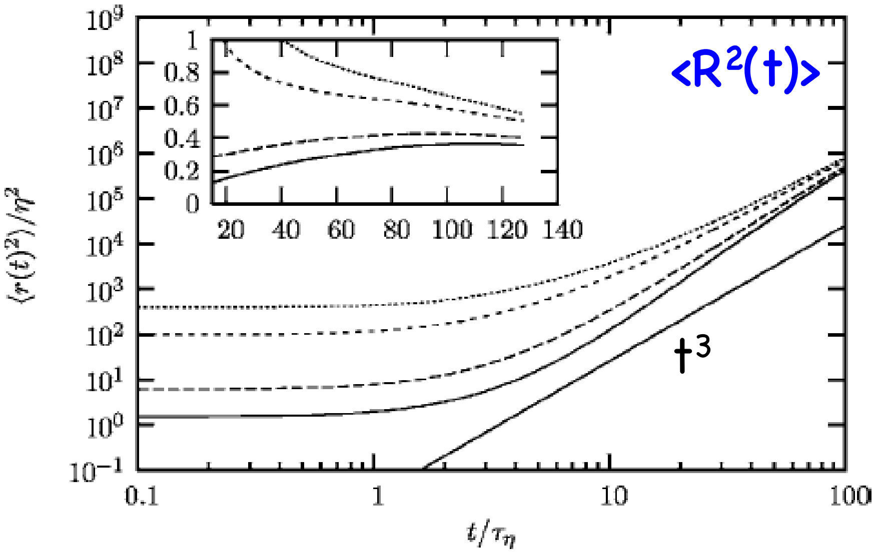


FIG. 1. The evolution of $\langle r(t)^2 \rangle / \eta^2$ vs t/τ_η for the initial separations $r_0 = 1.2\eta$, $r_0 = 2.5\eta$, $r_0 = 9.8\eta$, and $r_0 = 19.6\eta$. The straight line is proportional to t^3. Inset: $\langle r(t)^2 \rangle / \epsilon t^3$ for the same four initial separations starting from $t/\tau_\eta \sim 15$.

Fixed scale analysis

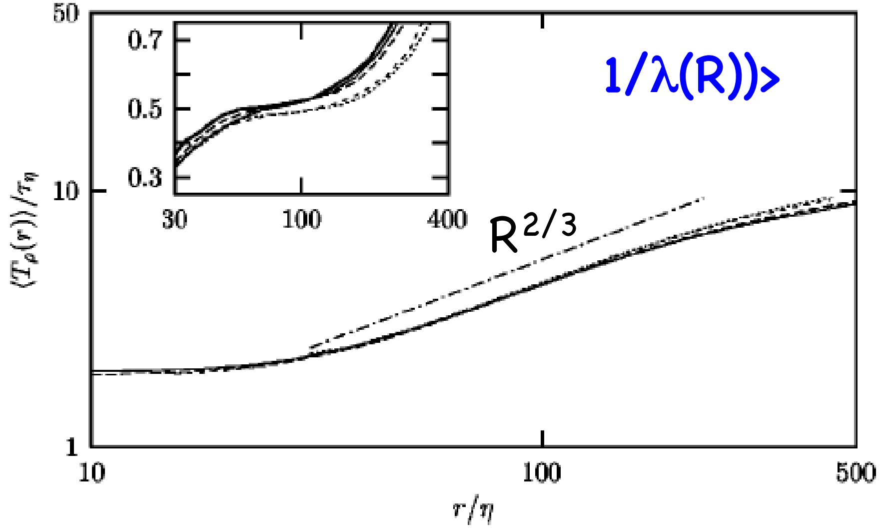
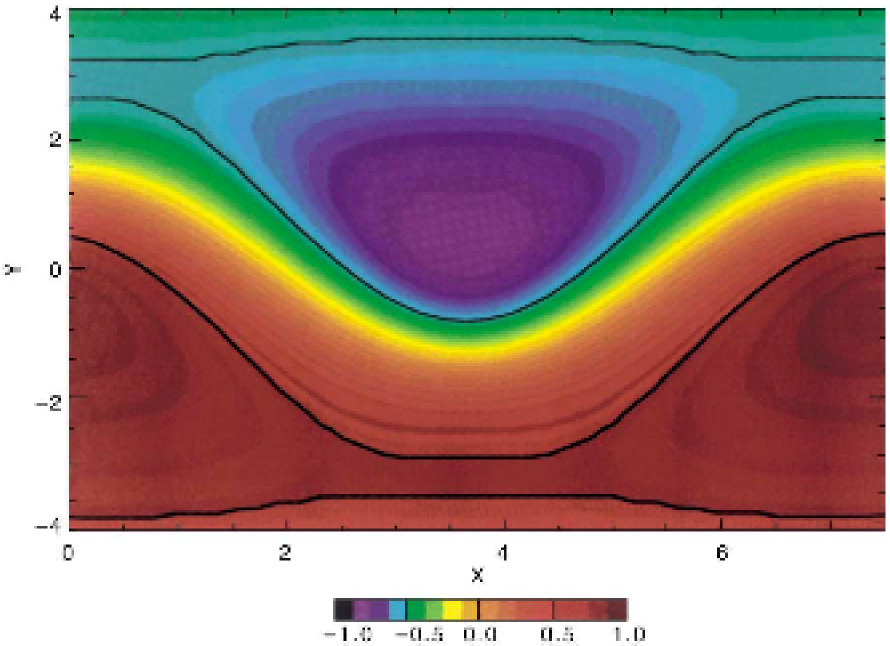


FIG. 5. The mean exit time for the initial separations $r_0 = 1.2\eta$ (thin continuous line), $r_0 = 2.5\eta$ (long dashed line), $r_0 = 9.8\eta$ (short dashed line), and $r_0 = 19.6\eta$ (dotted line) with $\rho = 1.25$. The straight line is proportional to $r^{2/3}$. In the inset we show Richardson's constant, g, vs r/η as given by (9) for the same initial separations at $R_\lambda = 284$. To evaluate the variability of g with the Reynolds number, we also plot a curve (thick continuous line) for the initial separation $r_0 = 1.2\eta$ at $R_\lambda = 183$.

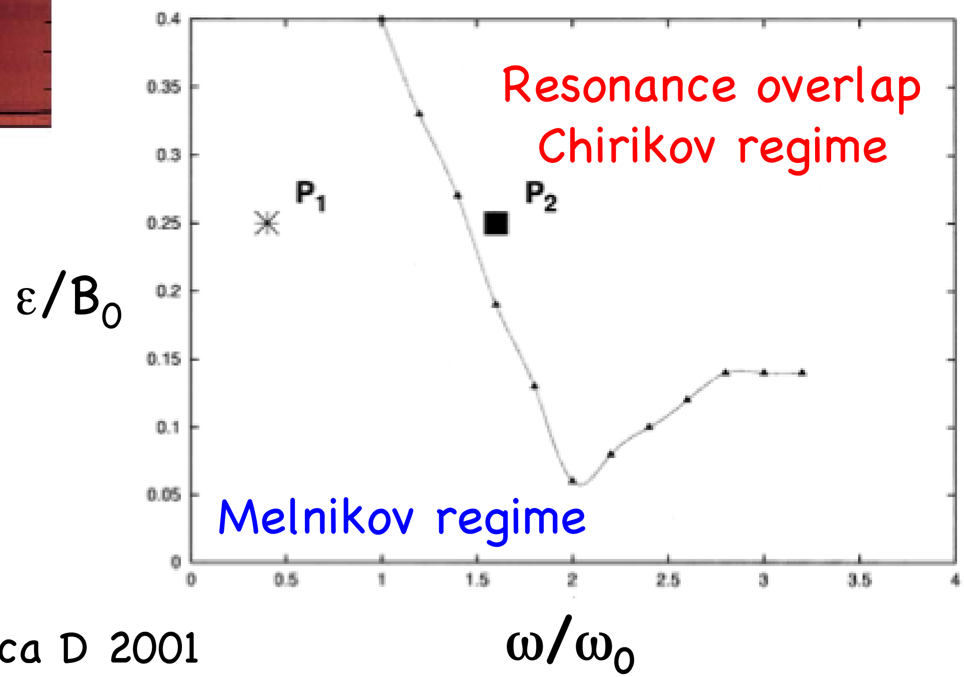
Biferale et al Phys. Fluids 2005

Detection of barriers to transport



$$\Psi = -\tanh\left[\frac{y - B\cos kx}{\sqrt{1 + k^2B^2\sin^2 kx}}\right] + cy,$$

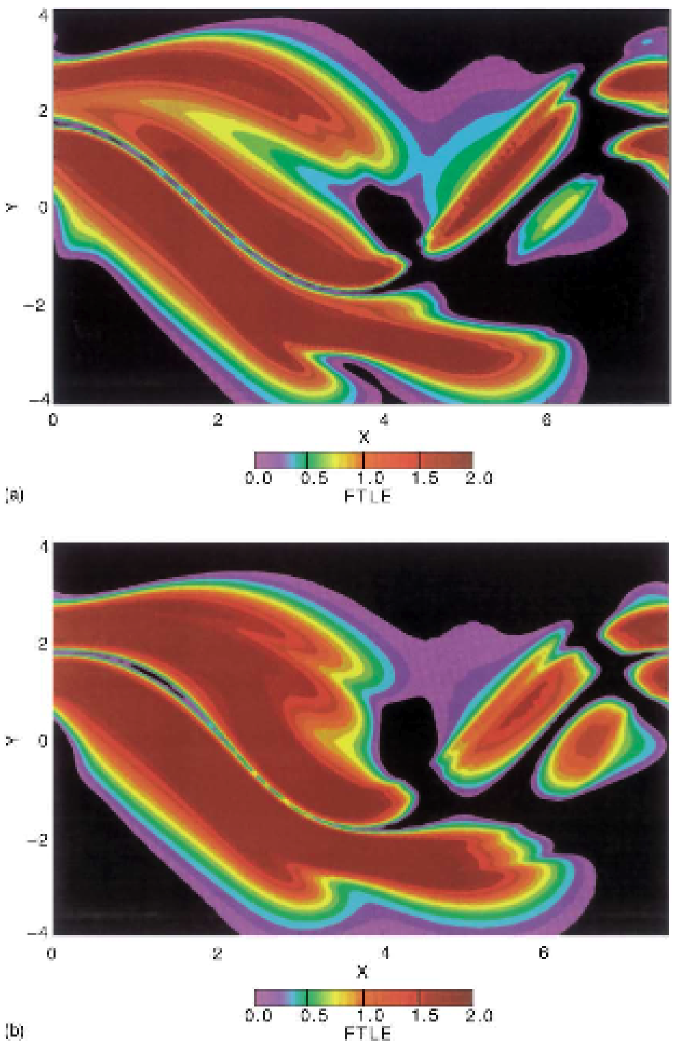
$$B = B_0 + \epsilon\cos(\omega t + \phi)$$



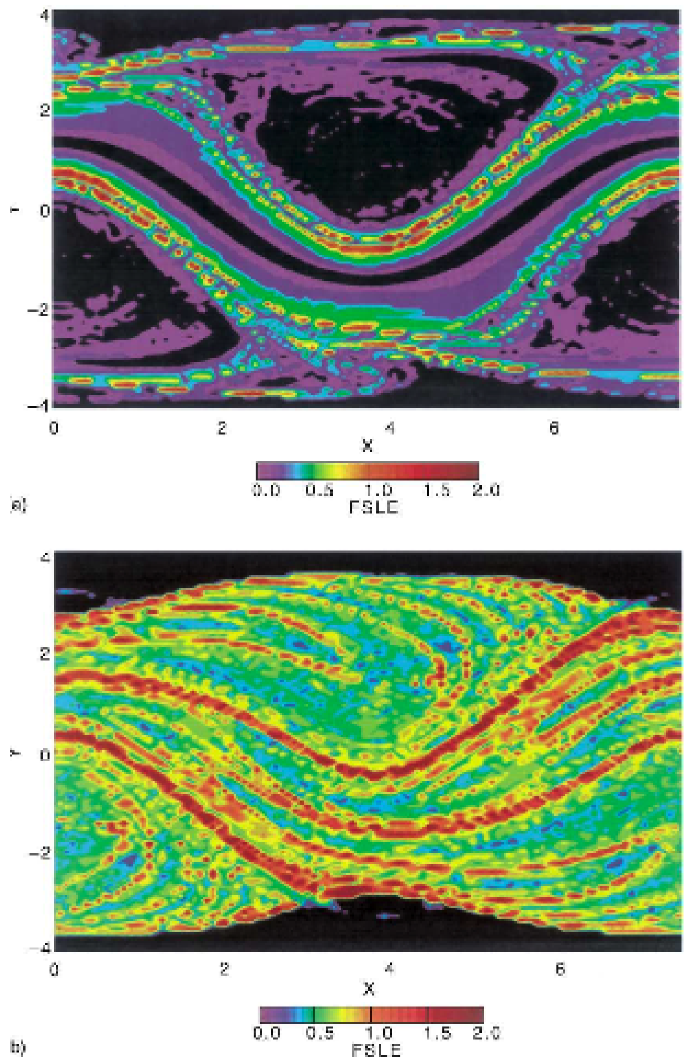
Vertical mixing
Vs
Vertial "localization"

Boffetta, Lacorata, Redaelli & AV Physica D 2001

FTLE vs FSLE



$$\gamma(\boldsymbol{x}, \delta, t)$$

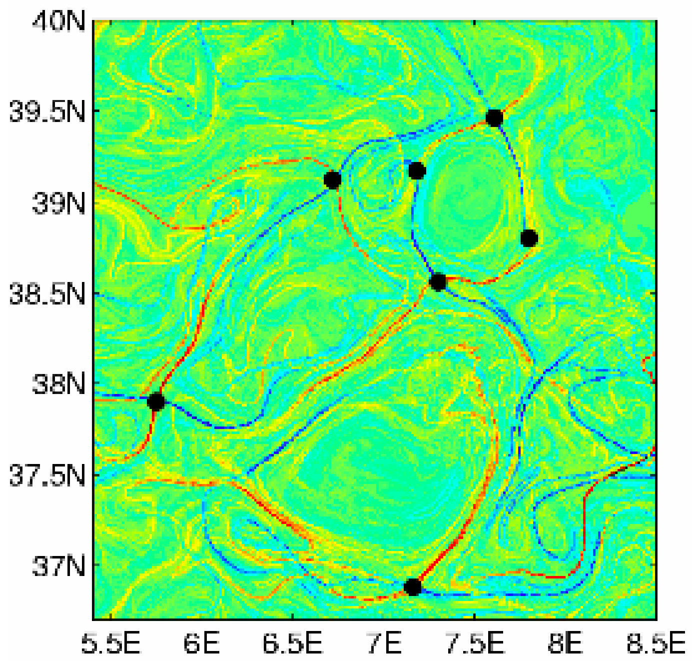


r=100

$$\gamma(\boldsymbol{x}, \delta, r\delta)$$

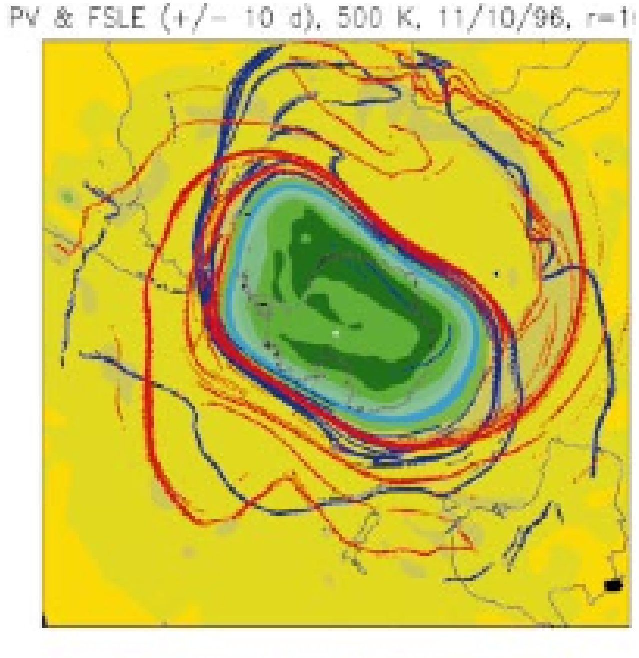
Forward/Backward FSLE as proxies for stable/unstable manifold

Forward/Backward FSLE



E. Henràndez-Garcia et al

Forward/Backward FSLE



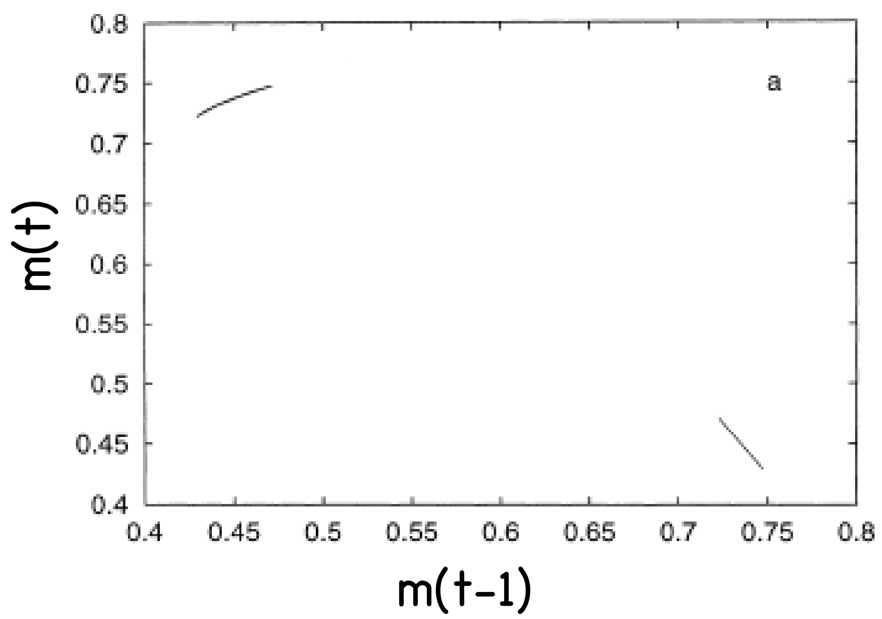
Koh & Legras Chaos 2002

Macroscopic/Collective chaos

$$x_n(t+1) = (1-\epsilon) f_a(x_n(t)) + \frac{\epsilon}{N}\sum_{i=1}^{N} f_a(x_i(t)),$$

Macroscopic observable $$m(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t),$$

Natural expectation would have been $m(t)=\langle m\rangle + O(N^{-1/2})$



<- but there are cases in which one observes a chaotic behavior for the global observable (Chaté & Manneville 1990/2)

Working hypothesis:

Microscopic<-> collisions FAST

Macroscopic<-> Hydrodynamics SLOW

Two time scales the "slowest" emerging From the dynamics

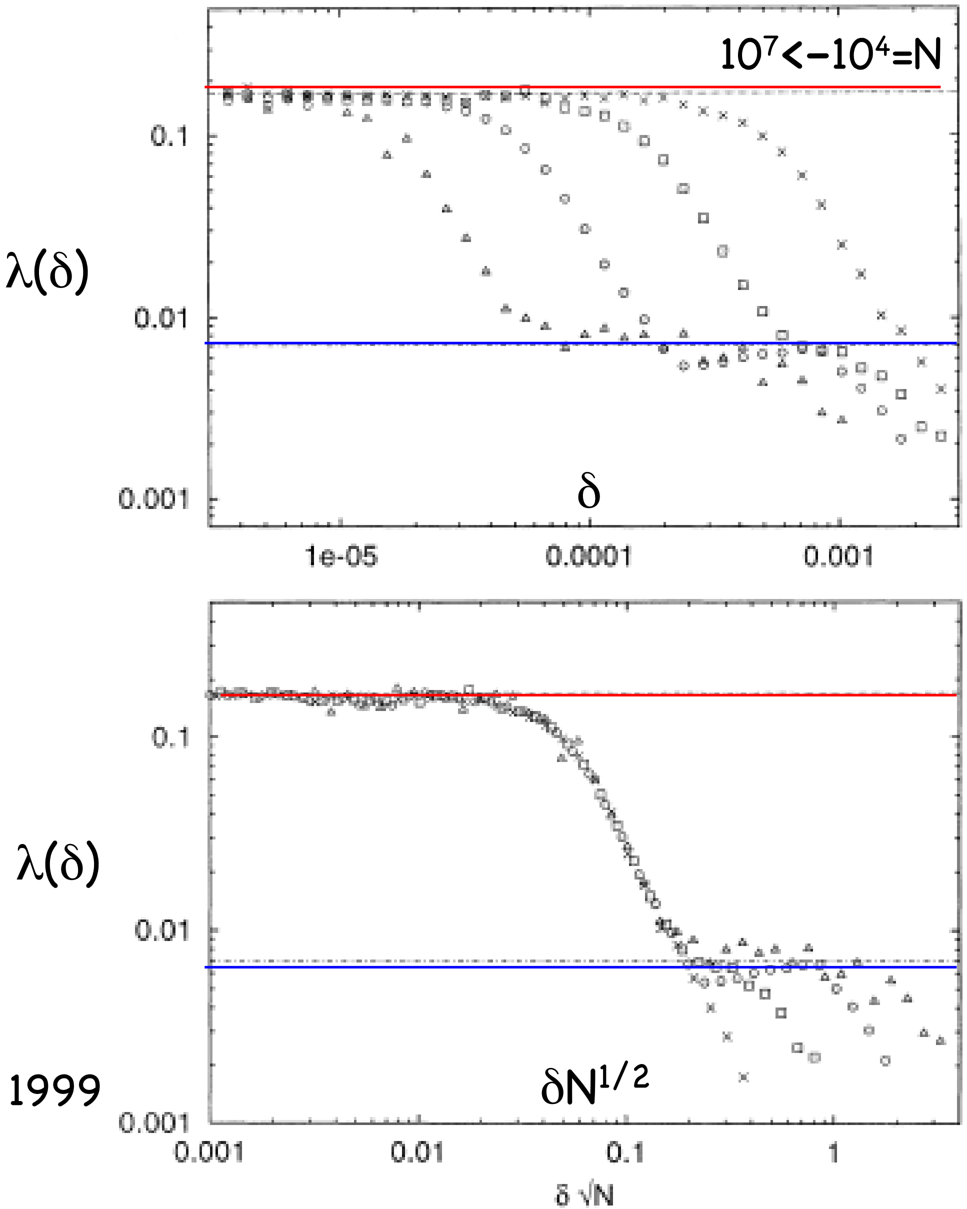
Macroscopic Chaos

$O(N^{-1/2})$

$$m(t+1) = F(m(t)) + \eta_N(t)$$

slow — fast

$$\delta = ||m - m'||$$

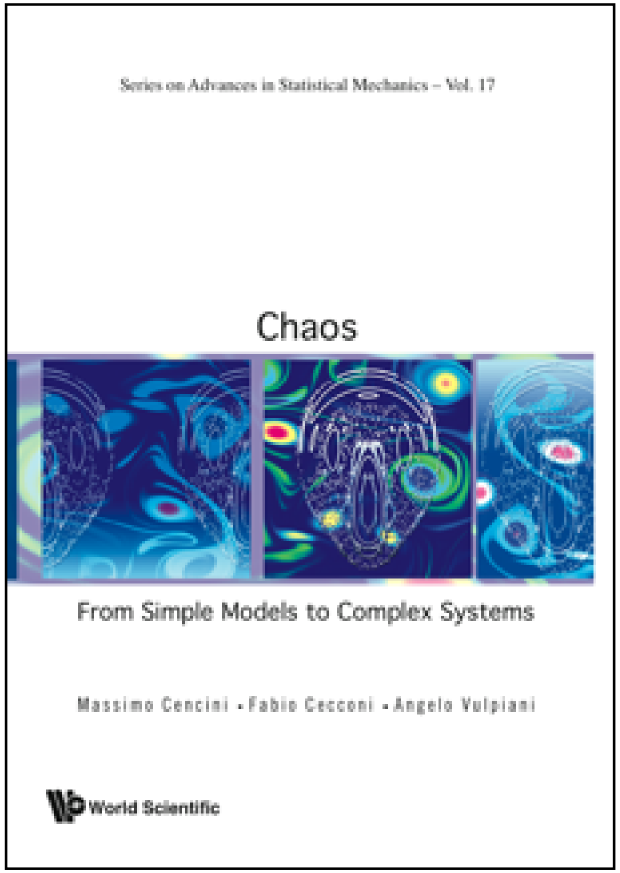


Shibata & Kaneko PRL 1998
Cencini, Falcioni, Vergni & AV Physica D 1999

Conclusions

- Though non-rigorous the FSLE demonstrated to be a useful tool to characterize predictability & transport properties in many practial problems
- Main advantages: free from the asymptotics of infinite time and infinitesimal perturbations which are unreachable and, often, irrelevant to practical situations. Rather easy to compute from data.
- Being based on (fixed) scale instead of (fixed) time is best suited to characterize those situations in which the different mechanisms at play manifest at different scales
- Word of care: necessity of physical input in the choice of the norm

References



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September 2009.
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(Review on application to predictability)

Boffetta, Celani, Cencini, Lacorata & AV Chaos **10**, 50 (2000)
(Review on application to transport)